Math 5112 - Lecture #11

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Last time: orthogonality relations for irreducible characters of finite groups

All vector spaces | algebras | repns are defined over $k = \mathbb{C}$ for this lecture

Let G be a finite group

The vector space of functions $G \to \mathbb{C}$ has form

$$(f_1, f_2) = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}$$

which is positive definite + Hermitian

Write $\mathrm{Irr}(G) = \{$ irreducible (complex) characters of $G\}$

Thm 1 $\mathrm{Irr}(G)$ is an orthonormal basis for the subspace of class functions G relative to the form $(\cdot,\cdot)$

maps $f: G \to \mathbb{C}$ with $f(gxg^{-1}) = f(x)\ \forall x, g \in G$

Cor If $\chi, \psi \in \mathrm{Irr}(G)$ then $(\psi, \chi) = \begin{cases} 1 & \text{if } \psi = \chi \\ 0 & \text{if } \psi \neq \chi \end{cases}$

Cor If χ, ψ are (not necessarily irreducible) characters of G then $(\psi, \chi) \in \{0, 1, 2, \ldots\}$

Pf We can decompose $\chi = \sum_{\phi \in \mathrm{Irr}(G)} a_{\chi\phi}\phi$ and $\psi = \sum_{\phi \in \mathrm{Irr}(G)} b_{\psi\phi}\phi$

where $a_{\chi\phi}, b_{\psi\phi}$ are nonnegative integers and $(\psi, \chi) = \sum_{\phi \in \mathrm{Irr}(G)} a_{\chi\phi} b_{\psi\phi}$ $\square$

Cor If (V, ρ_V) and (W, ρ_W) are finite dim. G-repns then $(\chi_V, \chi_W) = \dim \underbrace{\mathrm{Hom}_G(W, V)}$

space of linear maps $L: W \to V$ with $L \circ \rho_W(g) = \rho_V(g) \circ L \ \forall g \in G$

Pf If V and W are irreducible then

$$\dim \mathrm{Hom}_G(W, V) = \begin{cases} 1 & (V, \rho_V) \cong (W, \rho_W) \\ 0 & \text{else} \end{cases}$$ by Schur's lemma.

If the irred. decomp. of V and W are $V = \bigoplus_{i=1}^{r} m_i V_i$ and $W = \bigoplus_{j=1}^{s} n_j W_j$

then $\dim_G(W, V) = \sum_{i,j} m_i n_j \dim \mathrm{Hom}_G(W_j, V_i) = \sum_{i,j} m_i n_j (\chi_{W_j}, \chi_{V_i}) = (\chi_V, \chi_W)$

$\square$

Thm 2 For $g, h \in G$,

this is centralizer subgroup
$Z_g = \{x \in G \mid xg = gx\}$

$$\sum_{\psi \in \mathrm{Irr}(G)} \psi(g) \overline{\psi(h)} = \begin{cases} |Z_g| & \text{if } g \text{ and } h \text{ are conjugate in } G \\ 0 & \text{else} \end{cases}$$

Size of $\{xgx^{-1} \mid x \in G\}$ is $\dfrac{|G|}{|Z_g|}$

Character tables

Choose representatives $1 \overset{\text{by convention}}{=} g_1, g_2, g_3, \ldots, g_r$ of the distinct conjugacy classes $\{xgx^{-1} \mid x \in G\}$ in G

Let $\underbrace{\mathbb{1}}_{\text{let this denote the trivial character } G \to \{1\}} \overset{\text{by convention}}{=} \chi_1, \chi_2, \chi_3, \ldots, \chi_r$ be the elements of $\mathrm{Irr}(G)$

We call the matrix

	g_1	g_2	g_3	$\cdots$	g_r
χ_1	$\chi_1(g_1)$	$\chi_1(g_2)$	$\chi_1(g_3)$	$\cdots$	$\chi_1(g_r)$
χ_2	$\chi_2(g_1)$	$\chi_2(g_2)$	$\chi_2(g_3)$	$\cdots$	$\chi_2(g_r)$
χ_3	$\vdots$				$\vdots$
$\vdots$					
χ_r	$\chi_r(g_1)$	$\chi_r(g_2)$	$\chi_r(g_3)$	$\cdots$	$\chi_r(g_r)$

a <u>character table</u> for G

Suppose we are given this character table.

Thm 2 $\rightsquigarrow$ lets us compute $|G|$ and each $|Z_{g_i}|$

Namely: $|G| = |\chi_1(1)|^2 + |\chi_2(1)|^2 + |\chi_3(1)|^2 + \dots + |\chi_r(1)|^2 = |Z_1|$

$|Z_{g_i}| = |\chi_1(g_i)|^2 + |\chi_2(g_i)|^2 + |\chi_3(g_i)|^2 + \dots + |\chi_r(g_i)|^2$

Then $|K_{g_i}| = \dfrac{|G|}{|Z_{g_i}|}$ where $K_{g_i} = \{ x g_i x^{-1} \mid x \in G \}$

Thm 1 $\rightsquigarrow$ lets us compute the decomposition of any class function f on G as linear comb. of $\mathrm{Irr}(G)$.

Namely: $f = \sum_{i=1}^{r} (f, \chi_i)\chi_i$ and

$$(f, \chi_i) = \frac{1}{|G|} \sum_{g \in G} f(g)\chi_i(g) = \frac{1}{|G|} \sum_{j=1}^{r} f(g_j)\chi_i(g_j)\, \overbrace{|K_{g_j}|}^{\frac{|G|}{|Z_{g_j}|}}$$

$$= \sum_{j=1}^{r} \frac{\chi_i(g_j)}{|Z_{g_j}|} f(g_j)$$

<u>Ex</u> Suppose $G = S_3 = \{\text{group of permutations of } 1,2,3\}$

Character table is

	1	(12)	(123)
$\chi_{\yng(3)}$	1	1	1
$\chi_{\yng(2,1)}$	2	0	-1
$\chi_{\yng(1,1,1)}$	1	-1	1

$$|S_3| = |Z_1| = 1^2 + 2^2 + 1^2 = 6 = 3! \qquad (|S_n| = n!)$$

$$|Z_{(12)}| = 1^2 + 0^2 + (-1)^2 = 2$$

$$|Z_{(123)}| = 1^2 + (-1)^2 + 1^2 = 3$$

$$\Rightarrow |\mathcal{K}_1| = 1, \quad |\mathcal{K}_{(12)}| = 3, \quad |\mathcal{K}_{(123)}| = 2$$

It turns out that $K_1 = \{1\}$,

$$K_{(12)} = \{(12), (13), (23)\}$$

$$K_{(123)} = \{(123), (132)\}$$

Clearly $\chi_{\ydiagram{3}}\,\psi = \psi$ for any class function ψ

$$(\chi_{\ydiagram{1,1,1}})^2 = \chi_{\ydiagram{3}} \quad \text{and} \quad \chi_{\ydiagram{2,1}}\,\chi_{\ydiagram{1,1,1}} = \chi_{\ydiagram{2,1}}$$

But what is $(\chi_{\ydiagram{2,1}})^2$? Its values are

	1	(12)	(123)
$(\chi_{\ydiagram{2,1}})^2$	4	0	1

$$\left.\begin{aligned}
(\chi_{\ydiagram{3}}, (\chi_{\ydiagram{2,1}})^2) &= \frac{4}{6} + \frac{0}{2} + \frac{1}{3} = 1 \\
(\chi_{\ydiagram{2,1}}, (\chi_{\ydiagram{2,1}})^2) &= \frac{8}{6} + \frac{0}{2} + \frac{-1}{3} = 1 \\
(\chi_{\ydiagram{1,1,1}}, (\chi_{\ydiagram{2,1}})^2) &= \frac{4}{6} + \frac{0}{2} + \frac{1}{3} = 1
\end{aligned}\right\} \text{ so } \boxed{(\chi_{\ydiagram{2,1}})^2 = \chi_{\ydiagram{1,1,1}} + \chi_{\ydiagram{2,1}} + \chi_{\ydiagram{3}}}$$

Frobenius determinant — original motivation for repn thry.

G is a finite group (as usual)

Let $\{x_g\}_{g\in G}$ be commuting indeterminates indexed by G. Assume $|G| = n$ and list elements of G as

$$g_1 \quad g_2 \quad g_3 \quad \cdots \quad g_n$$

Let X_G be $n\times n$ matrix whose entry in position (i,j) is the variable $x_{g_i g_j}$. Call $\det(X_G) \in \mathbb{Z}[\,x_g \mid g\in G\,]$ the Frobenius determinant of G.

Fact Reordering the elements of G has no effect on value of $\det(\mathfrak{X}_G)$.

Pf Swapping g_i and g_j swaps rows i and j and then swaps columns of $\mathfrak{X}_G$, multiplying the determinant by $(-1)(-1) = +1$. $\square$

Thm If $|G|$ has exactly r conjugacy classes then the Frobenius determinant factors as

$$\det(\mathfrak{X}_G) = \prod_{j=1}^{r} \left(P_j(x) \right)^{\deg(P_j(x))}$$

where each $P_j(x)$ is an irreducible polynomial in $\mathbb{C}[x_g \mid g \in G]$ and $P_i(x)$ does not divide $P_j(x)$ if $i \neq j$.

Ex Suppose $G = \mathbb{Z}/2\mathbb{Z} \cong \{\pm 1\}$

Then $$\Phi_G = \begin{array}{cc} \begin{array}{cc} +1 & -1 \end{array} & \\ \begin{bmatrix} x_+ & x_- \\ x_- & x_+ \end{bmatrix} & \begin{array}{c} +1 \\ -1 \end{array} \end{array}$$

So $\det(\Phi_G) = x_+^2 - x_-^2 = (x_+ - x_-)(x_+ + x_-)$.

G has 2 conjugacy classes $\{1\}$ and $\{-1\}$.

Ex Suppose $G = \mathbb{Z}/3\mathbb{Z} = \{1 = c^3, c, c^2\}$

has 3 conjugacy classes $\{1\}, \{c\}, \{c^2\}$.

Let $y_i = x_{c^{i-1}}$. Then

$$\mathbb{X}_G = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{ccc} 1 & c & c^2 \\ \end{array}$$

$$\mathbb{X}_G = \begin{bmatrix} y_1 & y_2 & y_3 \\ y_2 & y_3 & y_1 \\ y_3 & y_1 & y_2 \end{bmatrix}\begin{array}{l} 1 \\ c \\ c^2 \end{array}$$

$$\Rightarrow \det(\mathbb{X}_G) = y_1(y_2y_3 - y_1^2) - y_2(y_2^2 - y_1y_3) + y_3(y_1y_2 - y_3^2)$$

$$= 3y_1y_2y_3 - y_1^3 - y_2^3 - y_3^3$$

$$= (-y_1 - y_2 - y_3)(y_1 + \theta y_2 + \theta^2 y_3)(y_1 + \theta^2 y_2 + \theta y_3)$$

where $\theta = e^{2\pi\sqrt{-1}/3}$ is a primitive 3^{rd} root of unity

Goal: prove thm about how $\det(X_G)$ factorizes using repn theory, following Frobenius's original argument from ~1896. Will need a lemma:

Lemma Suppose y_{ij} for $1 \le i,j \le n$ are distinct commuting indeterminates. Then $\det[y_{ij}]_{1\le i,j\le n}$ is an irreducible polynomial in $\mathbb{C}[y_{ij} \mid 1 \le i,j \le n]$

Pf $\det[y_{ij}]$ is homogeneous polynomial, so if it's not irreducible then it must factor as a product of homogeneous irreducible polynomials

In a homogeneous polyn., setting some variables to zero either gives zero or a homogeneous polynomial.

Thus if $\det[y_{ij}]$ is reducible then setting some variables to zero gives zero or another reducible polynomial.

But setting $y_{ij} = 0$ for all $(i,j) \notin \left\{ \begin{matrix} (1,1), (1,2) \\ (2,2), (2,3) \\ (3,3), (3,4) \\ \vdots \\ (n,n), (n,1) \end{matrix} \right\}$

transforms

$$\det[y_{ij}] \rightsquigarrow \det \begin{bmatrix} y_{11} & y_{12} & & & \\ & y_{22} & y_{23} & & \\ & & \ddots & \ddots & \\ & & & \ddots & y_{n-1,n} \\ y_{n1} & & & & y_{nn} \end{bmatrix} = \underbrace{y_{11} y_{22} \cdots y_{nn} \pm y_{12} y_{23} \cdots y_{n1}}_{\text{Clearly irreducible, nonzero since two monomials involve disjoint variables}}$$

$\square$

Pf of theorem about $\det(X_G)$

Let $L(x) = \sum_{g \in G} x_g \, g \in \mathbb{C}[x_g \mid g \in G][G] \stackrel{\text{def}}{=} R$

If $h \in G$ then $L(x)h = \sum_{g \in G} x_g \, gh = \sum_{k \in G} x_{kh^{-1}} k$

$\Rightarrow$ matrix of $L(x) : R \to R$ is $[x_{kh^{-1}}]_{(h,k) \in G \times G}$

which is formed from X_G by permuting some columns

$\Rightarrow \det(X_G) = \pm \det(L(x))$

So can just factor $\det(L(x))$.

By Masohke's theorem,

$$\mathbb{C}[G] = \bigoplus_{i=1}^{r} (V_i)^{\oplus \dim(V_i)}$$

where $V_1, V_2, \ldots, V_r$ are irreducible subrepns with r = #conjugacy classes in G.

So $$\det(L(x)) = \prod_{i=1}^{r} \Big(\underbrace{\det\big(L(x)|_{V_i}}_{\text{call this } P_i(x)}\big)\Big)^{\dim(V_i)}$$

Now we just need to argue that $P_i(x)$'s are irreducible and pairwise non-proportional

To see this, choose a positive-def. Hermitian G-invariant form on each V_i, then choose an orthonormal basis $\{e_{ij}\}$ for V_i, then let $\{E_{ijk}\}$ be the corresponding matrix elements $G \to \mathbb{C}$ defined by $E_{ijk}(g) = (\rho_{V_i}(g) e_{ij}, e_{ik})$

Matrix of $L(x)|_{V_i}$ in basis $\left\{ \sum_{g \in G} E_{ijk}(g) g \right\}_{j,k}$

is $\left[\underbrace{\sum_{g \in G} x_g E_{ijk}(g)}_{\text{call this } y_{ijk}} \right]_{1 \le j,k \le \dim V_i}$

By lemma, $\det [y_{ijk}]_{1 \le j,k \le \dim V_i}$ is irreducible, so same is true of $P_i(x)$. As functions of the y_{ijk}'s these polynomials involve disjoint variables, so they are non-proportional. $\square$

Going between the x_g and y_{ijk} variables does not affect irreducibility or proportionality since the latter variables are a homogeneous linear transformation of the x_g's.