Math 5112 - Lecture #25

Math 5112 - Lecture #25

Course recap:

Choose an algebraically closed field K

An algebra over K is a K-vector space A with a bilinear multiplication. Unless otherwise stated, "algebra" means "unital associative algebra"

Saying "(V, ρ) is a representation of A" means

V is a vector space and $\rho : A \to \mathrm{End}(V)$ is an algebra morphism

Saying "V is a representation of A" means V is a left A-module.

Morphisms $\phi : (V_1, \rho_1) \to (V_2, \rho_2)$ of A-reps are linear maps $\phi : V_1 \to V_2$ with

$$\begin{tikzcd} V_1 \arrow[r, "\phi"] \arrow[d, "\rho_1(a)"'] & V_2 \arrow[d, "\rho_2(a)"] \\ V_1 \arrow[r, "\phi"] & V_2 \end{tikzcd}$$

commuting $\forall a \in A$

For groups, Lie algebras, quivers, etc., we have other notions of representations, but these are all equivalent to certain full subcategories of algebra representations

We have notions of subrepresentations and irreducible representations, as well as direct sums of representations and indecomposable representations.

Every irreducible is indecomposable, not vice versa.

Schur's lemma Given a nonzero morphism of repns

$$\phi : (V_1, \rho_1) \to (V_2, \rho_2)$$

① ϕ is $\cong$ if both V_i irreducible (even if k not alg. closed)

② ϕ is scalar map if $(V_1, \rho_1) = (V_2, \rho_2)$ is irreducible and finite-dim. (requires k to be alg. closed)

Cor. Commutative algebras over alg. closed fields have only 1-d irreducible repns.

Given vector spaces $V_1, V_2, V_3, \ldots, V_k$ (and

Form the tensor product vector space

$$V_1 \otimes V_2 \otimes \cdots \otimes V_k \leftarrow \text{dim is } \prod_{i=1}^{k} \dim V_i$$

Tensor algebra of $V = \bigoplus_{n \geq 0} \underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ times}} = TV$

Quotients of TV: symmetric algebra SV

$= TV / \langle x \otimes y - y \otimes x \mid x, y \in V \rangle$

exterior algebra ΛV

$= TV / \langle x \otimes x \mid x \in V \rangle$

univ. envel. alg. (when V is Lie alg.)

If V is (A,B)-bimodule and W is (B,C)-bimodule for algebras A, B, C then $V \otimes_B W$ is a (A,C)-bimodule.

If V is a right A-module and W is a left A-module then $V \otimes_B W$ is just a vector space

In both cases $V \otimes_B W$ as a vector space is a quotient of $V \otimes W$ by $\langle xb \otimes y - x \otimes by \mid \forall x \in V, y \in W, b \in B \rangle$

Def A repn of A is semisimple or completely reducible if it is isomorphic to a direct sum of irreducible representations

Assume A is an algebra with $\dim A < \infty$.

Def The radical of A is the set of elements in A that act as zero in every irreducible repn of A.

Fact $\mathrm{Rad}(A)$ is the largest nilpotent 2-sided ideal

Suppose $A = \bigoplus_{i=1}^{r} \mathrm{Mat}_{d_i}(k)$ for some $d_1, d_2, \dots, d_r > 0$

Convenient to view $A \subset \mathrm{Mat}_n(k)$ for $n = d_1 + d_2 + \dots + d_r$

<u>Thm</u> For each index i, A has an irreducible representation $V_i \cong K^{d_i}$ (as vector spaces) and every finite-dimensional repn of A is a direct sum of copies of $V_1, V_2, \dots, V_r$

And in this case $\mathrm{Rad}\, A = 0$

<u>Thm</u> Any finite dimensional algebra A has finitely many irreducible representations $V_1, V_2, \dots, V_r$ up to isomorphism, each V_i has finite dimension, and

$$A / \mathrm{Rad}(A) \cong \bigoplus_{i=1}^{r} \mathrm{End}(V_i) \cong \bigoplus_{i=1}^{r} \mathrm{Mat}_{d_i}(k)$$

where $d_i = \dim(V_i)$

Each $\mathrm{End}(V_i)$ has dimension $d_i^2 = \dim(V_i)^2$ so

<u>Cor</u> If $\dim A < \infty$ then $\dim A - \dim \mathrm{Rad}(A) = \sum_{i=1}^{r} \dim(V_i)^2 \leq \dim A$

Def. A finite-dimensional algebra A is called semisimple if $\mathrm{Rad}(A) = 0$.

Prop. Assume A is an algebra $/ k$ with $\dim A < \infty$.

The following are equivalent:

① A is semisimple

② $\sum_{i=1}^{r} \dim(V_i)^2 = \dim A$ where $V_1, V_2, \ldots, V_r$ are the distinct isomorphism classes of irreducible A-reps

③ $A \cong \bigoplus_{i=1}^{r} \mathrm{Mat}_{d_i}(k)$ for some $d_1, d_2, \ldots, d_r > 0$

④ Any finite-dim repn of A is semisimple

⑤ The regular repn of A is semisimple

Let (V,ρ) be a finite-dimensional repn of A.

The <u>character</u> of (V,ρ) is the linear map $\chi_{(V,\rho)}: A \to k$ with the formula $\chi_{(V,\rho)}(a) = \operatorname{trace}(\rho(a))$ for $a \in A$.

Say that a character $\chi_{(V,\rho)}$ is <u>irreducible</u> if (V,ρ) is irreducible

<u>Thm</u> Assume A is semisimple and $\dim A < \infty$. Then the irreducible characters of A are a basis for $\left(A/[A,A]\right)^*$

linear maps $A/[A,A] \to k$

$\boxed{\text{Jordan-Hölder thm}}$: If V is an A-repn with $\dim V < \infty$ then there exists a filtration

$$0 = V_0 \subset V_1 \subset \dots \subset V_n = V$$

where each V_i is a subrepn, each V_i / V_{i-1} is irreducible, and any other filtration with these properties has same length n and the same quotients V_i / V_{i-1} (up to isomorphism and permutations of indices).

$\boxed{\text{Krull-Schmidt thm:}}$ If V is an A-repn with $\dim V < \infty$ then there exists a decomposition $V \cong \bigoplus_{i \in I} V_i$ where each V_i is indecomposable and this decomp. is unique up to isomorphism and rearrangement of factors.

Important special case: Consider $V^{\oplus n}$ where V is already irreducible.

Repns of tensor products:

If A and B are k-algebras then so is $A \otimes B$ in a natural way: $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$ (vector space tensor product)

you can tensor A-repns to get an A-repn if you have a morphism $A \to A \otimes A$

If V is an A-repn and W is a B-repn then $V \otimes W$ is an $A \otimes B$-repn.

Thm If V, W are irreducible and finite-dimensional then so is $V \otimes W$ (as an $A \otimes B$-repn).

Up to $\cong$, all irreducible finite-dim. repns of $A \otimes B$ arise in this way.

Ex If G is finite group $\cong H \times I$ then $k[G] \cong k[H] \otimes k[I]$

A representation of a group G is an (algebra) repn (V, ρ) of the group algebra $k[G]$.

This means that $\rho(k[G]) \subseteq \mathrm{End}(V)$ all linear maps $V \to V$

$\rho(G) \subseteq GL(V)$ invertible linear maps $V \to V$

What makes group repns more interesting than the (trivial) repn theory of semisimple algebras is the distinguished basis of group elements.

Assume $\boxed{G \text{ is a finite group}}$

$\boxed{\text{Maschke's theorem}}$ The group algebra $k[G]$ is semisimple if and only if char(k) does not divide $|G|$.

(means all irreducible G-repns are finite-dim, and all finite-dim. G-repns are direct sums of irr. repns)

Assume (V, ρ) is a fin. dim. G-repn.
Then its character is the linear map $\chi_{(V,\rho)} : k[G] \to k$
with $g \longmapsto \operatorname{trace}(\rho(g))$ for $g \in G$.

Say that $\chi_{(V,\rho)}$ is irreducible if (V,ρ) is.

Let $\boxed{\mathrm{Irr}(G)}$ denote set of irreducible characters of G. Some things that always hold:

1. If $(V,\rho) \cong (V',\rho')$ then $\chi_{(V,\rho)} = \chi_{(V',\rho')}$
2. Each $\chi = \chi_{(V,\rho)}$ is a class function on G, meaning a map $G \to K$ that is constant on conjugacy classes $\Longleftrightarrow$ $\chi(ghg^{-1}) = \chi(h)$ for all $g,h \in G$

When $K[G]$ is semisimple, the following holds:

3. $\mathrm{Irr}(G)$ is a basis for vector space of class functions on G

4. If $\mathrm{char}(k) = 0$, then $\chi_{(V,\rho)} = \chi_{(V',\rho')}$ if and only if $(V,\rho) \cong (V',\rho')$. Doesn't hold if $\mathrm{char}(k) > 0$.

5. $$\sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 = |G|$$

Assume $K = \mathbb{C}$.

The vector space of functions $G \to \mathbb{C}$ has form

$$(f_1, f_2) = \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)}$$ which is $\boxed{\text{positive definite + Hermitian}}$

Thm $\operatorname{Irr}(G)$ is an orthonormal basis for the subspace of class functions on G relative to the form $(\cdot,\cdot)$

Cor If $\chi, \psi \in \operatorname{Irr}(G)$ then $(\psi, \chi) = \begin{cases} 1 & \text{if } \psi = \chi \\ 0 & \text{if } \psi \neq \chi \end{cases}$

Cor If χ, ψ are (not necessarily irreducible) characters of G then $(\psi, \chi) \in \{0, 1, 2, \dots\}$

this is centralizer subgroup

$Z_g = \{ x \in G \mid xg = gx \}$

<u>Thm</u> For $g, h \in G$,

$$\sum_{\psi \in \mathrm{Irr}(G)} \psi(g)\overline{\psi(h)} = \begin{cases} |Z_g| & \text{if } g \text{ and } h \text{ are conjugate in } G \\ 0 & \text{else} \end{cases}$$

in this case

$|Z_g| = |Z_h|$

Size of $\{ xgx^{-1} \mid x \in G \}$ is $\dfrac{|G|}{|Z_g|}$

Character tables

Choose representatives $1 \overset{\text{by convention}}{=} g_1, g_2, g_3, \dots, g_r$ of the distinct conjugacy classes $\{xgx^{-1} \mid x \in G\}$ in G

Let $\underbrace{\mathbb{1}}_{\text{let this denote the trivial character } G \to \{1\}} \overset{\text{by convention}}{=} \chi_1, \chi_2, \chi_3, \dots, \chi_r$ be the elements of $\mathrm{Irr}(G)$

We call the matrix

$$\begin{array}{c|ccccc} & g_1 & g_2 & g_3 & \cdots & g_r \\ \hline \chi_1 & \chi_1(g_1) & \chi_1(g_2) & \chi_1(g_3) & \cdots & \chi_1(g_r) \\ \chi_2 & \chi_2(g_1) & \chi_2(g_2) & \chi_2(g_3) & \cdots & \chi_2(g_r) \\ \chi_3 & \vdots & & & & \vdots \\ \vdots & & & & & \\ \chi_r & \chi_r(g_1) & \chi_r(g_2) & \chi_r(g_3) & \cdots & \chi_r(g_r) \end{array}$$

a <u>character table</u> for G

Frobenius-Schur indicator of V (irreducible rep.)

Let $\varepsilon(V) = \varepsilon(\chi_V) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } V \text{ is real type} \\ 0 & \text{if } V \text{ is complex type} \\ -1 & \text{if } V \text{ is quaternionic type} \end{cases}$

Thm $\boxed{\varepsilon(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2)}$ for any $\chi \in \mathrm{Irr}(G)$

Cor $\boxed{\# \{g \in G \mid g^2 = 1\} = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\, \varepsilon(\chi)}$ $\leq \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)$

call these the involutions of G

this is equality for many nice groups like S_n, or Coxeter groups

<u>Thm</u> The dimension of V divides $|G|$

(when V is an irreducible complex repn of finite group G)

<u>Thm</u> If G is finite of order $p^a q^b$ for distinct primes $p, q > 1$ and integers $a, b > 0$, then G is <u>solvable</u> (ie has normal composition series with abelian quotients).

If H is a subgroup of a finite group G and V is a G-repn, then we write $\boxed{\mathrm{Res}_H^G(V)}$ to denote V viewed as an H-repn via restriction.

Character of $\mathrm{Res}_H^G(V)$ is $\chi_V|_H$ (character restricted to H)

Dimension of $\mathrm{Res}_H^G(V)$ is $\dim(V)$

Call $\mathrm{Res}_H^G(V)$ the <u>restriction</u> of V.

Suppose W is an H-repn where $H \subset G$ is a subgroup

Let $\mathrm{Ind}_H^G(W) = \left\{ f : G \to W \mid f(hx) = \rho_W(h) f(x) \ \forall h \in H, x \in G \right\}$

① $\cong \underbrace{k[G]}_{\substack{\text{viewed} \\ \text{as } (k[G],\, k[H])\text{-bimodule}}} \otimes_{k[H]} \underbrace{W}_{\substack{\text{viewed as} \\ \text{left-}k[H]\text{-module}}} \longrightarrow \sum_i \underbrace{g_i \otimes w_i}_{\substack{= g_i h \otimes \rho_W(h^{-1}) w_i \\ \forall h \in H}}$

$\mathrm{Ind}_H^G(W)$ is a G-repn (called the induced repn) for action

$$g \cdot f : x \longmapsto f(xg) \quad \text{for } g \in G,\ f \in \mathrm{Ind}_H^G(W).$$

Character of $\mathrm{Ind}_H^G(W)$ is $\boxed{g \longmapsto \sum_{\substack{i \in \{1,2,\dots,r\} \\ g_i^{-1} g g_i \in H}} \chi_W(g_i^{-1} g g_i)}$ ②

where $g_1, g_2, \dots, g_r$ are a complete set of coset representatives for $G/H = \{gH \mid g \in G\}$

Write $\mathrm{Res}_H^G(\chi_V) \overset{\text{def}}{=} \chi_V|_H$ and $\mathrm{Ind}_H^G(\chi_W)$ for the characters of $\mathrm{Res}_H^G(V)$ and $\mathrm{Ind}_H^G(\chi_W)$.

__Cor__ Assuming $k = \mathbb{C}$, if χ is any character of G and ψ is any character of H, then

$$\left(\chi, \mathrm{Ind}_H^G(\psi)\right)_G = \left(\mathrm{Res}_H^G(\chi), \psi\right)_H$$

$$= \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\mathrm{Ind}_H^G(\psi)(g)} \qquad = \frac{1}{|H|} \sum_{h \in H} \chi(h) \overline{\psi(h)}$$

$\rightsquigarrow$ Appl: can develop a criteria for when $\mathrm{Ind}_H^G(\psi)$ is irreducible that only involves restricting to smaller subgroups then inducing to H

Further topics

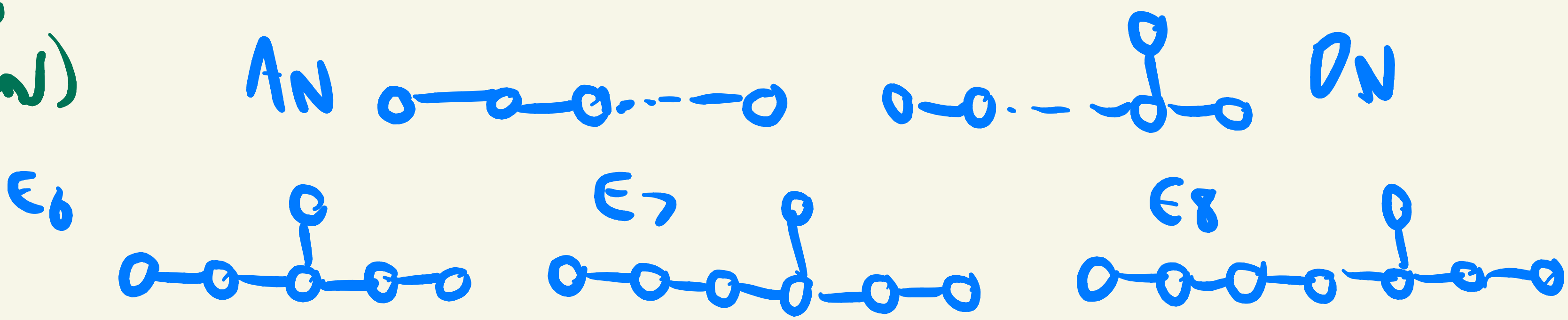
① Constructing the irreducible representations of the symmetric group S_n and their characters:

- $\mathrm{Irr}(S_n)$ indexed by partitions $\lambda \vdash n$
- To each $\lambda \vdash n$ we construct a Specht module V^λ with character χ_λ.
- [Product] formula for $\dim V^\lambda$ and even for $\chi_\lambda(C_\mu)$

② Indecomparable quiver representations and Gabriel's thm.

↳ a finite quiver has finitely many $\cong$ classes of indecomparable reps if and only if its connected components ignoring edge orientations are each

Recall: quiver repr assigns vector spaces to each vertex and linear maps to each arrow (no conditions)



③ Category theory : a (second) introduction

- object
- morphisms
- small category
- enriched category
- full subcategory
- abelian category
- functor
- natural transformations
- adjoints
- Yoneda's lemma and representability

etc...

④ Homological algebra + projectives,
Ext, Tor, Cohomology, chain complexes
exact functors, Morita equivalence, blocks

Assume $\dim A < \infty$, k alg. closed, (not nec. semisimple)

Suppose $M_1, M_2, M_3, \ldots, M_n$ are a complete list of non-isomorphic irreducible A-modules.

Thm For each i, there is a unique-up-to-isomorphism indecomposable, finitely generated, projective A-module P_i such that $\dim \mathrm{Hom}_A(P_i, M_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

(Call P_i the projective cover of M_i.) And it holds that...

Moreover, it holds that $\boxed{A \cong \bigoplus_{i=1}^{n} \underbrace{(\dim M_i)}_{} P_i}$ and $P_1, P_2, P_3, \ldots, P_n$ are a complete list of non-isomorphic, finitely generated indecomposable, projective A-modules.

$= P_i \oplus P_i \oplus \cdots \oplus P_i$
$\dim M_i$ summands

(when A is semisimple then $M_i \cong P_i \ \forall i$)