

MIDTERM SOLUTIONS – MATH 2121, FALL 2026

Problem 1. (10 points) A country, which is led by a **president**, has an autonomous region, which is led by a **governor**, and a city, which is led by a **mayor**.

The annual salary of the president is P .

The annual salary of the governor is G .

The annual salary of the mayor is M .

The country has an old saying “一馬九熊不如九龍” which has been turned into a law mandating that the mayor’s salary plus 9 times the president’s salary must be equal to the governor’s salary.

However, it is reported that the amount that the mayor earns in two years is 1 million HKD more than the governor earns in 1 year.

This year the budget to pay all three salaries is 9 million HKD.

Write down a **linear system** that characterizes P , G , and M . Then **solve** for these values using HKD for units.

Solution. The three conditions give the linear system

$$\begin{array}{l} M + 9P = G, \\ 2M = G + 1000000, \\ P + G + M = 9000000. \end{array} \quad \text{or} \quad \begin{array}{l} 9P - G + M = 0, \\ -G + 2M = 1000000, \\ P + G + M = 9000000. \end{array}$$

The first equation gives $G = 9P + M$.

Substituting this into the second equation, we obtain $M - 9P = 1000000$. Thus

$$M = 9P + 1000000 \quad \text{and} \quad G = 18P + 1000000.$$

Substituting into the third equation gives $28P + 2000000 = 9000000$. Therefore the annual salaries, in HKD, are

$$P = 250,000, \quad G = 5,500,000, \quad M = 3,250,000.$$

Alternatively, using 1,000,000 HKD as units, the augmented matrix row reduces to

$$\begin{aligned} \left[\begin{array}{ccc|c} 9 & -1 & 1 & 0 \\ 0 & -1 & 2 & 1 \\ 1 & 1 & 1 & 9 \end{array} \right] &\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & -1 & 2 & 1 \\ 9 & 0 & -1 & -1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & -28 & -91 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 4 & 13 \end{array} \right] \\ &\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 3.25 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & 1 & 0 & 5.5 \\ 0 & 0 & 1 & 3.25 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0.25 \\ 0 & 1 & 0 & 5.5 \\ 0 & 0 & 1 & 3.25 \end{array} \right] \end{aligned}$$

so $P = 250,000$, $G = 5,500,000$, and $M = 3,250,000$.

Coincidentally:

The annual salary of the President of China is $\approx 171,600$ HKD (136,620 CNY)

The annual salary of the Chief Executive of Hong Kong is $\approx 5,630,000$ HKD.

The annual salary of the Chief Executive of Macau is $\approx 3,142,400$ HKD.

Problem 2. (10 points) Let $\text{nnz}(A)$ be the number of nonzero entries in a matrix A . For example, if $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ then $\text{nnz}(A) = 4$.

Let $\mathcal{S}$ be the set of 2×3 matrices **whose entries are each 0 or 1 and which are in reduced echelon form**. Define

$$f(n, r) = |\{A \in \mathcal{S} : \text{nnz}(A) = n, \text{rank}(A) = r\}|.$$

Enter the values of $f(n, r)$ in the following table. When $f(n, r) = 0$ you can leave the corresponding entry of the table **blank**.

Solution. The completed table is

$f(n, r)$	$r = 0$	$r = 1$	$r = 2$	$r = 3$
$n = 0$	1	0	0	0
$n = 1$	0	3	0	0
$n = 2$	0	3	3	0
$n = 3$	0	1	3	0
$n = 4$	0	0	1	0
$n = 5$	0	0	0	0
$n = 6$	0	0	0	0

Rank 0. The zero matrix is the only possibility, so $f(0, 0) = 1$.

Rank 1. The second row must be zero. The first row can be any nonzero 0–1 row of length 3, since its first nonzero entry is 1. Thus $f(n, 1) = \binom{3}{n}$ for $n = 1, 2, 3$.

Rank 2. The possible pivot columns are $(1, 2)$, $(1, 3)$, and $(2, 3)$. The matrices are

$$\begin{bmatrix} 1 & 0 & s \\ 0 & 1 & t \end{bmatrix}, \quad \begin{bmatrix} 1 & s & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where each free entry can be 0 or 1. The first family contributes 1, 2, 1 matrices with 2, 3, 4 nonzero entries, respectively. The second contributes one with 2 and one with 3 nonzero entries. The third contributes one with 2 nonzero entries. Hence

$$f(2, 2) = 3, \quad f(3, 2) = 3, \quad f(4, 2) = 1.$$

All other entries are zero. There are $1 + 7 + 7 = 15$ matrices in $\mathcal{S}$ in total.

Problem 3. (10 points) Find all values of $a, b \in \mathbb{R}$ for which the linear system

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_1 + ax_2 + x_3 &= 2 \\ 2x_1 + (a+1)x_2 + (a-1)x_3 &= b \end{aligned}$$

has

- (1) a unique solution,
- (2) no solution,
- (3) infinitely many solutions.

Solution.

Subtract the sum of the original first and second rows from the third row, and then subtract the first row from the second row. This gives

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & a & 1 & 2 \\ 2 & a+1 & a-1 & b \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & a-1 & 0 & 0 \\ 0 & 0 & a-3 & b-4 \end{array} \right].$$

- (1) There is a unique solution precisely when

$$\boxed{a \notin \{1, 3\}, \quad b \in \mathbb{R}.}$$

In this case there is a pivot in each of the first three columns.

- (2) There is no solution precisely when

$$\boxed{a = 3, \quad b \neq 4.}$$

In this case the last column is a pivot column.

- (3) There are infinitely many solutions precisely when

$$\boxed{a = 1, \, b \in \mathbb{R} \quad \text{or} \quad a = 3, \, b = 4.}$$

For $a = 1$, the second row is zero and the last row determines x_3 . For $a = 3$ and $b = 4$, the last row is zero and the second row determines x_2 . In each case the last column is not a pivot and there is one free variable.

Problem 4. (10 points) This problem has two parts.

The vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ shown below have equal y -coordinates.

(a) Draw the region that contains all vectors of the form

$$a\mathbf{u} + b\mathbf{v} \quad \text{for } a, b \in \mathbb{R} \text{ with } 1 \leq a \leq 2 \text{ and } 1 \leq b \leq 2.$$

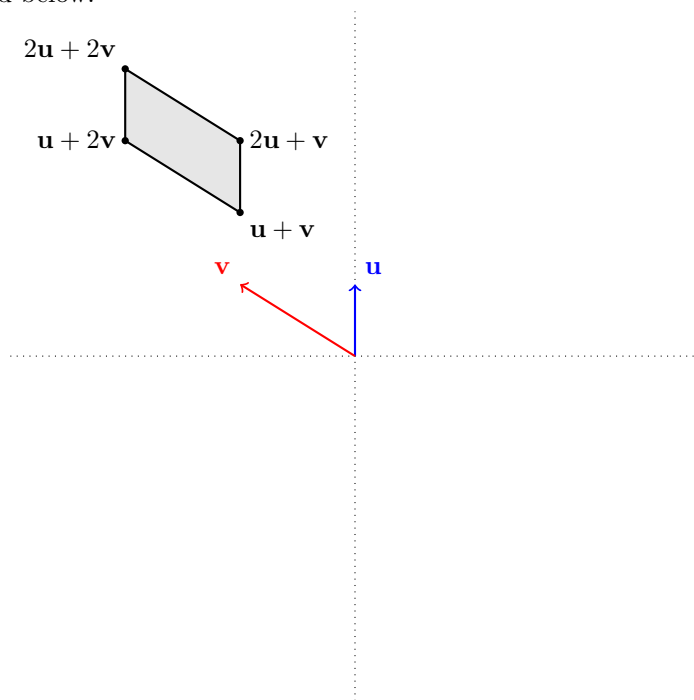
Solution. Write $a = 1 + s$ and $b = 1 + t$, where $0 \leq s, t \leq 1$. Then

$$a\mathbf{u} + b\mathbf{v} = (\mathbf{u} + \mathbf{v}) + s\mathbf{u} + t\mathbf{v}.$$

The region is the **filled parallelogram, including its boundary**, with vertices

$$\mathbf{u} + \mathbf{v}, \quad 2\mathbf{u} + \mathbf{v}, \quad 2\mathbf{u} + 2\mathbf{v}, \quad \mathbf{u} + 2\mathbf{v}.$$

It is shaded below.



Problem 4 (continued).

(b) Suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ are vectors such that $\begin{bmatrix} \mathbf{x} & \mathbf{y} \end{bmatrix} \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The area of the region in part (a) is $\frac{1}{2}$. Draw $\mathbf{x}$ and $\mathbf{y}$ below:

Solution. As shown in the given diagram, write

$$\mathbf{u} = \begin{bmatrix} 0 \\ h \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} -w \\ h \end{bmatrix}, \quad h, w > 0.$$

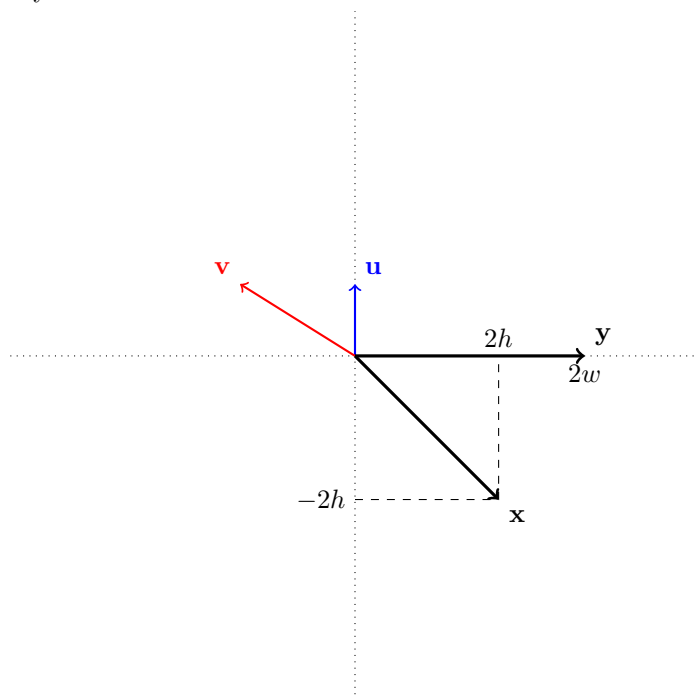
The region in part (a) is a translate of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$. Its area is $|\det \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}| = wh$, so $wh = \frac{1}{2}$. Therefore

$$\begin{bmatrix} \mathbf{x} & \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -w \\ h & h \end{bmatrix}^{-1} = \frac{1}{wh} \begin{bmatrix} h & w \\ -h & 0 \end{bmatrix} = \begin{bmatrix} 2h & 2w \\ -2h & 0 \end{bmatrix}.$$

Thus

$$\mathbf{x} = \begin{bmatrix} 2h \\ -2h \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 2w \\ 0 \end{bmatrix}.$$

Draw $\mathbf{x}$ to the right and downward by twice the height of $\mathbf{u}$, and draw $\mathbf{y}$ horizontally to the right by twice the horizontal distance from the vertical axis to the tip of $\mathbf{v}$.



Problem 5. (10 points) This problem has two parts. Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

(a) Find a basis for $\text{Nul}(A)$.

Solution. Row reduction gives

$$\begin{aligned} A &\xrightarrow{R_3 \leftarrow R_3 - R_1 - R_2} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & 0 \end{bmatrix} \\ &\xrightarrow{R_3 \leftarrow -\frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ &\xrightarrow{\substack{R_1 \leftarrow R_1 - R_3 \\ R_2 \leftarrow R_2 - R_3}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \end{aligned}$$

Thus $Ax = 0$ is equivalent to

$$x_1 = -x_4, \quad x_2 = x_4, \quad x_3 = 0.$$

Writing $x_4 = t$, every vector in the null space has the form

$$x = t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Since the displayed vector is nonzero, a basis is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Problem 5 (continued).

(b) Explain why

$$H = \{x \in \mathbb{R}^4 : Ax \in \mathbb{R}\text{-span}\{b\}\}$$

is a subspace of $\mathbb{R}^4$ and find a basis for H .

Solution. First, $0 \in H$, because $A0 = 0$. If $x, y \in H$, then $Ax = sb$ and $Ay = tb$ for some $s, t \in \mathbb{R}$. For any $\alpha, \beta \in \mathbb{R}$, linearity gives

$$A(\alpha x + \beta y) = \alpha Ax + \beta Ay = (\alpha s + \beta t)b \in \mathbb{R}\text{-span}\{b\}.$$

Therefore $\alpha x + \beta y \in H$, so H is a subspace.

To find a basis, observe that b is the third column of A . Thus

$$Ae_3 = b, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

A vector x is in H exactly when $Ax = tb$ for some $t \in \mathbb{R}$, or equivalently,

$$A(x - te_3) = 0.$$

By part (a), this means

$$x = s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{for some } s, t \in \mathbb{R}.$$

The two displayed vectors are linearly independent: in a linear combination equal to zero, the fourth coordinate forces $s = 0$, and the third coordinate then forces $t = 0$. Hence a basis for H is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Problem 6. (10 points) For which values of $a \in \mathbb{R}$ is the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & a & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

invertible? For all such values of a , find a formula for A^{-1} .

Solution. Row reduction gives

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & a & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2-a & -1 & 1 & 1-a \end{array} \right].$$

The first three columns can only be converted to the identity matrix by further row operations if $a \neq 2$. Thus A is invertible if and only if $a \neq 2$.

Assume $a \neq 2$. Then divide the third row by $2 - a$, subtract the third row from the second, and then subtract the second row from the first. The result is

$$A^{-1} = \frac{1}{a-2} \begin{bmatrix} a-1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & a-1 \end{bmatrix}.$$

Problem 7. (10 points) For parameters $a, b \in \mathbb{R}$, we seek a linear transformation

$$T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

satisfying all four conditions

$$\begin{aligned} T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, & T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \\ T\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right) &= \begin{bmatrix} a \\ 1 \\ 2 \end{bmatrix}, & T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) &= \begin{bmatrix} 1 \\ 1 \\ b \end{bmatrix}. \end{aligned}$$

Determine all values of a and b for which such a transformation exists. Find the standard matrix of T in every case where it exists.

Solution. Since

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

linearity requires

$$\begin{bmatrix} a \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Thus $a = 1$ is necessary.

Now suppose $a = 1$ and let b be any real number. The images of the standard basis vectors are forced by the given equations:

$$\begin{aligned} T(e_1) &= \begin{bmatrix} 1 \\ 1 \\ b \end{bmatrix}, \\ T(e_2) &= T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) - T(e_1) = \begin{bmatrix} 0 \\ -1 \\ 1-b \end{bmatrix}, \\ T(e_3) &= T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) - T(e_2) = \begin{bmatrix} 0 \\ 2 \\ b \end{bmatrix}. \end{aligned}$$

The linear map with these three column vectors satisfies all four equations. Consequently the allowed parameters and the unique standard matrix are

$$a = 1, \quad b \in \mathbb{R}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 2 \\ b & 1-b & b \end{bmatrix}.$$

Problem 8. (10 points) Let A be a 2×3 matrix and B a 3×2 matrix, with

$$\text{rank}(A) = \text{rank}(B) = 2.$$

Determine all possible ordered pairs

$$(\text{rank}(AB), \text{rank}(BA)).$$

For each possible pair, give matrices A and B that attain it. Justify that no other pairs are possible.

Solution. The map $A : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is onto, so

$$\text{Col}(BA) = \{B(Ax) : x \in \mathbb{R}^3\} = \{By : y \in \mathbb{R}^2\} = \text{Col}(B).$$

Hence $\text{rank}(BA) = 2$ in every case.

Since AB is 2×2 , its rank is at most 2. If its rank were zero, then $AB = 0$, so $\text{Col}(B) \subseteq \text{Nul}(A)$. This is impossible because

$$\dim \text{Col}(B) = 2 \quad \text{but} \quad \dim \text{Nul}(A) = 3 - \text{rank}(A) = 1.$$

Thus $\text{rank}(AB)$ must be 1 or 2, and the only possible pairs are

$$(1, 2) \quad \text{and} \quad (2, 2).$$

Both possibilities occur, as the following examples show.

The pair (1, 2). Take

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Both matrices have rank 2, and

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

which have ranks 1 and 2, respectively.

The pair (2, 2). Take

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Both matrices have rank 2, and

$$AB = I_2, \quad BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which both have rank 2.

Problem 9. (10 points) Let $F, M \in \mathbb{R}$. Consider the vectors

$$u = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 6 \\ F \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} M \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$$

Suppose $A = I_5 + uv^\top$ (where I_5 is the 5×5 identity matrix). Compute $\det(A)$.

Solution. We first show directly that

$$\det(I_n + uv^\top) = 1 + v^\top u.$$

Let $e_1, \dots, e_n$ be the standard basis vectors. The j th column of $I_n + uv^\top$ is $e_j + v_j u$. By multilinearity in the columns, its determinant expands into terms obtained by choosing either e_j or $v_j u$ in each column.

The term choosing every e_j is $\det(I_n) = 1$. Any term choosing $v_j u$ in two or more columns is zero, because those columns are linearly dependent. If only column j uses $v_j u$, its contribution is

$$v_j \det(e_1, \dots, e_{j-1}, u, e_{j+1}, \dots, e_n) = v_j u_j.$$

Indeed, after expanding $u = \sum_i u_i e_i$, every summand except $u_j e_j$ produces a repeated column. Summing all the surviving terms proves the identity.

For the vectors in this problem,

$$v^\top u = M \cdot 2 + 2 \cdot 0 + 1 \cdot 2 + 2 \cdot 6 + 1 \cdot F = 2M + F + 14.$$

Therefore

$$\boxed{\det(A) = 2M + F + 15.}$$