

MIDTERM – MATH 2121, FALL 2024

Name:

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Student ID:

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Email:

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Section: L1

L2

Tutorial: T1A

T1B

T1C

T1D

T2A

T2B

T2C

T2D

Problem #	Points Possible	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
Total	90	

You have **120 minutes** to complete this exam.

No books, notes, or electronic devices can be used on the test.

You are encouraged to draw a

 around your final answers.

You may lose points if graders cannot determine your answer to a question.

Partial credit can be given on some problems if you show your work. Good luck!

Problem 1. (10 points)

Find the general solution to the linear system

$$\begin{cases} x_1 + 3x_2 - 5x_3 = 4 \\ x_1 + 4x_2 - 8x_3 = 7 \\ -3x_1 - 7x_2 + 9x_3 = -6. \end{cases}$$

Problem 2. (10 points)

Compute how many 2×4 matrices A exist with all of the following properties:

- each entry of A is equal to 0 and 1,
- A is in reduced echelon form, and
- A is the augmented matrix of a consistent linear system.

Justify your answer.

Problem 3. (10 points)

Suppose $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a linear transformation satisfying

$$T\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

We are told that at least 2 of the columns of the standard matrix of T are equal to each other as vectors. Find all possibilities for the standard matrix of T .

Problem 4. (10 points)

Let $a_1, a_2, \dots, a_n$ be any real numbers. Define A to be the lower-triangular $n \times n$ matrix whose entry in position (i, j) is a_j whenever $i \geq j$. This means A looks like

$$A = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ a_1 & a_2 & 0 & & 0 \\ a_1 & a_2 & a_3 & & 0 \\ \vdots & & & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix}.$$

Determine when A is invertible and describe A^{-1} .

Be as explicit as possible and justify your answer.

Problem 5. (10 points)

Consider the three matrices

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -4 & 5 & 8 & -4 \\ 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \end{bmatrix}, \quad C = \begin{bmatrix} -7 & 1 & -1 & 6 & -3 \\ -1 & 3 & 2 & -2 & 1 \\ -4 & 8 & 5 & -4 & 2 \end{bmatrix}.$$

The first matrix A has reduced echelon form

$$\text{RREF}(A) = \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This question has three parts:

- (a) Find a basis for $\text{Col}(A)$, a basis for $\text{Nul}(A)$, and the rank of A .

- (b) Find a basis for $\text{Col}(B)$, a basis for $\text{Nul}(B)$, and the rank of B .

- (c) Find a basis for $\text{Col}(C)$, a basis for $\text{Nul}(C)$, and the rank of C .

Problem 6. (10 points)

Many algorithms in this course extract information about a matrix A from properties of its reduced echelon form $\text{RREF}(A)$.

Explain how you can use $\text{RREF}(A)$ to answer the following 5 questions.

Write your answers in the form “YES if (... some condition ...), NO otherwise”.

(a) Are the columns of A linearly independent?

(b) Is A the coefficient matrix of any inconsistent linear system?

(c) Is A invertible?

(d) Is A row equivalent to another matrix B ?

(e) Does A have full rank?

(A matrix is said to have *full rank* if its rank is as large as possible among all matrices that share its size.)

Problem 7. (10 points) Let $A = \begin{bmatrix} 2 & 4 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 7 & 2 \\ 2 & 1 \end{bmatrix}$.

Find a basis for the intersection of $\text{Col}(A)$ and $\text{Col}(B)$.

Problem 8. (10 points) Alice and Bob play the following game: they start with an empty 4×4 matrix A and take turns writing numbers in each of the 16 positions.

Once the matrix A is filled, Bob wins if $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \in \text{Nul}(A)$ and Alice wins otherwise.

If Alice goes first, then which player (if any) has a winning strategy?

What is this strategy?

Problem 9. (10 points) What is the greatest common divisor of the numbers that occur as $\det(A)$ as A ranges over all invertible 4×4 matrices whose entries are each equal to $+1$ or -1 ?

The *greatest common divisor* of a set of nonzero integers is defined to be the largest positive integer that divides every integer in the set.

Justify your answer.

