This document is an **exact transcript** of the lecture, with extra summary and vocabulary sections. Due to time constraints, the lectures sometimes only contain limited illustrations, proofs, and examples. For a more thorough discussion of the course content, **consult the textbook**.

Summary

Quick summary of today's notes. Lecture starts on next page.

Linear independence:

- A list of vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ is *linearly independent* if

$$v_i \notin \mathbb{R}\text{-span}\{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_p\} \text{ for all } i = 1, 2, 3, \dots, p.$$

  This happens precisely when the only way to express

$$0 = c_1 v_1 + c_2 v_2 + \dots + c_p v_p$$

  for $c_1, c_2, \dots, c_p \in \mathbb{R}$ is by taking $c_1 = c_2 = \dots = c_p = 0$.

- If the vectors are not linearly independent, then they are *linearly dependent*.
- Two vectors are linearly dependent if one is a scalar multiple of the other.
- A list of vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ is linearly independent if and only if the $n \times p$ matrix

$$A = \left[\begin{array}{cccc} v_1 & v_2 & \dots & v_p \end{array}\right]$$

  has a pivot in every column.

- Any sufficiently long list of vectors is linearly dependent.

  More precisely, a list of vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ with $p > n$ is always linearly dependent.

Functions and linearity:

- Writing $f : X \to Y$ means that f is a function that transforms inputs $x \in X$ to outputs $f(x) \in Y$.

  The set X is called the *domain* while Y is called the *codomain* of f.

- Let m, n be positive integers. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function then the following mean the same thing:
  - For any $u, v \in \mathbb{R}^n$ and $c \in \mathbb{R}$ it holds that $f(u+v) = f(u) + f(v)$ and $f(c \cdot v) = c \cdot f(v)$.
  - There exists an $m \times n$ matrix A such that $f(v) = Av$ for all $v \in \mathbb{R}^n$.

  Such functions f are said to be *linear*. The matrix A is called the *standard matrix* of f.

- Every linear function $f : \mathbb{R}^n \to \mathbb{R}^m$ has exactly one standard matrix.
- If $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear then its standard matrix is $A = \left[\begin{array}{cccc} f(e_1) & f(e_2) & \dots & f(e_n) \end{array}\right]$ where

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n, \quad \dots \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^n.$$

1 Last time: multiplying vectors and matrices

Given a matrix $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ and a vector $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ we define

$$\boxed{Av = v_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + v_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \in \mathbb{R}^m.}$$

We refer to Av as the product of A and v, or the vector given by multiplying v by A.

Example. We have $\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 5 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 6 \end{bmatrix} + \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} -1+0+3 \\ -5+0+7 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

If A is an $m \times n$ matrix and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $b \in \mathbb{R}^m$, then we call $\boxed{Ax = b}$ a *matrix equation*.

A matrix equation $Ax = b$ has the same solutions as the linear system with augmented matrix $\begin{bmatrix} A & b \end{bmatrix}$.

Theorem. Let A be an $m \times n$ matrix. The following are equivalent:

1. $Ax = b$ has a solution for any $b \in \mathbb{R}^m$.
2. The span of the columns of A is all of $\mathbb{R}^m$.
3. A has a pivot position in every row.

Example. The matrix equation

$$\begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

may fail to have a solution since

$$\mathsf{RREF}\left(\begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$$

has pivot positions only in rows 1 and 2.

2 Linear independence

In this section we are going to discuss a property of **lists** of vectors that we call *linear independence*.

Let us first explain what this term means for lists of zero, one, two, or three vectors.

(0) The empty list of zero vectors is always defined to be *linearly independent*.

(1) A list containing just one vector $v_1 \in \mathbb{R}^n$ is defined to be *linearly independent* if

$$v_1 \neq 0.$$

(2) A list containing two vectors $v_1, v_2 \in \mathbb{R}^n$ is defined to be *linearly independent* if

$$\text{there is no } a \in \mathbb{R} \text{ with } v_1 = av_2 \quad \text{AND} \quad \text{there is no } b \in \mathbb{R} \text{ with } v_2 = bv_1, \qquad (*)$$

meaning that **neither vector is a scalar multiple of the other**. We can rewrite (*) as

$$v_1 \notin \mathbb{R}\text{-span}\{v_2\} \quad \text{AND} \quad v_2 \notin \mathbb{R}\text{-span}\{v_1\}.$$

(3) A list containing three vectors $v_1, v_2, v_3 \in \mathbb{R}^n$ is defined to be *linearly independent* if

$$v_1 \notin \mathbb{R}\text{-span}\{v_2, v_3\} \quad \text{AND} \quad v_2 \notin \mathbb{R}\text{-span}\{v_1, v_3\} \quad \text{AND} \quad v_3 \notin \mathbb{R}\text{-span}\{v_1, v_2\}.$$

Definition. In general, a list of p vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ is defined to be *linearly independent* if

$$v_i \notin \mathbb{R}\text{-span}\{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_p\} \text{ for all } i = 1, 2, 3, \dots, p.$$

When $p = 1$ we interpret "$\mathbb{R}$-span{}" to mean the set $\{0\}$ containing just the zero vector $0 \in \mathbb{R}^n$.

Example. It is easy to tell if lists of one or two vectors are linearly independent.

(a) The single vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is linearly independent, but $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not.

(b) The two vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ are linearly independent, but $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix}$ are not.

(c) What about the three vectors $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$? Since

$$v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 2v_1 + v_3 \in \mathbb{R}\text{-span}\{v_1, v_3\}$$

the list v_1, v_2, v_3 is **not** linearly independent.

It is not clear how we could prove that three or more vectors actually are linearly independent.

For this, it is useful to reformulate our definition of linear independence in the following way.

Proposition. A list of vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ is linearly independent if and only if the equation

$$x_1v_1 + x_2v_2 + \cdots + x_pv_p = \begin{bmatrix} v_1 & v_2 & \dots & v_p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = 0 \qquad (**)$$

has no other solutions besides the trivial solution $x_1 = x_2 = \cdots = x_p = 0$.

A nontrivial solution to (**) is called a *linear dependence* among the vectors $v_1, v_2, \dots, v_p$.

Proof. Suppose our vectors are **not** linearly independent, so that we can write

$$v_i = a_1 v_1 + \dots + a_{i-1}v_{i-1} + a_{i+1}v_{i+1} + \dots + a_p v_p \in \mathbb{R}\text{-span}\{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_p\}$$

for some index $i \in \{1, 2, \dots, p\}$ and some scalars $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_p \in \mathbb{R}$. Then

$$(x_1, \dots, x_{i-1}, \quad x_i, \quad x_{i+1}, \dots, x_p) = (a_1, \dots, a_{i-1}, \quad -1, \quad , a_{i+1}, \dots, a_p)$$

is a nontrivial solution to (**), since even if the a's are all zero the value $x_i = -1$ is nonzero.

Conversely, suppose (**) has a nontrivial solution $(c_1, c_2, \dots, c_p)$.

Then one of the numbers c_i in this solution must be nonzero. But then

$$0 = c_1 v_1 + c_2 v_2 + \dots + c_p v_p$$

and we can divide both sides by the nonzero number c_i to get

$$0 = \tfrac{1}{c_i}\left(c_1 v_1 + c_2 v_2 + \dots + c_p v_p\right) = \tfrac{c_1}{c_i} v_1 + \dots + \tfrac{c_{i-1}}{c_i} v_{i-1} + v_i + \tfrac{c_{i+1}}{c_i} v_{i+1} + \dots + \tfrac{c_p}{c_i} v_p.$$

This means that

$$v_i = -\tfrac{c_1}{c_i} v_1 - \dots - \tfrac{c_{i-1}}{c_i} v_{i-1} - \tfrac{c_{i+1}}{c_i} v_{i+1} - \dots - \tfrac{c_p}{c_i} v_p \in \mathbb{R}\text{-span}\{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_p\}$$

so the given list of vectors is **not** linearly independent. $\square$

Corollary. The columns of a matrix A are linearly independent if and only if A has a pivot position in every column.

Proof. Suppose the columns of A are $v_1, v_2, \dots, v_p \in \mathbb{R}^n$.

By the previous theorem, these vectors are linearly independent if and only if the matrix equation $Ax = 0$ has exactly one solution (the trivial solution $x = 0$).

This happens precisely when A has a pivot position in every column so that there are no free variables. $\square$

How to determine if $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ are linearly independent.

- Form the $n \times p$ matrix $A = \begin{bmatrix} v_1 & v_2 & \dots & v_p \end{bmatrix}$.
- Reduce A to echelon form to find its pivot columns.
- If every column of A is a pivot column, then the vectors are linearly independent.

  If some column of A is not a pivot column, then the vectors are not linearly independent.

Example. The vectors $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$, and $\begin{bmatrix} 5 \\ 9 \\ 15 \end{bmatrix}$ are linearly independent, since

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 9 \\ -1 & 5 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 9 \\ 0 & 7 & 20 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathsf{RREF}(A).$$

Every column of A contains a pivot position, so the columns of A are linearly independent.

3 Linear dependence

A list of vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ is *linearly dependent* if it is not linearly independent. For example:

(0) The empty list of zero vectors is never *linearly dependent*.

(1) A list containing just one vector $v_1 \in \mathbb{R}^n$ is *linearly dependent* if $v_1 = 0$.

(2) A list containing two vectors $v_1, v_2 \in \mathbb{R}^n$ is *linearly dependent* if

$$\text{there is some } a \in \mathbb{R} \text{ with } v_1 = av_2 \quad \text{OR} \quad \text{there is some } b \in \mathbb{R} \text{ with } v_2 = bv_1,$$

meaning that **one vector is a scalar multiple of the other**.

(3) An arbitrary list $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ is *linearly dependent* if some

$$v_i \in \mathbb{R}\text{-span}\{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_p\}.$$

This occurs precisely when a linear dependence exists among the vectors, or equivalently when

$$\begin{bmatrix} v_1 & v_2 & \dots & v_p \end{bmatrix}$$

has at least one non-pivot column.

Example. The vectors $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$, and $\begin{bmatrix} 5 \\ 9 \\ 16 \end{bmatrix}$ are linearly dependent since

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 9 \\ -1 & 5 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 9 \\ 0 & 7 & 21 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \mathsf{RREF}(A)$$

and the last matrix has no pivot position in column 3.

Proposition. Suppose $v_1, v_2, \dots, v_p \in \mathbb{R}^n$. If $p > n$ then the list of vectors is linearly dependent.

Proof. The $n \times p$ matrix $A = \begin{bmatrix} v_1 & v_2 & \dots & v_p \end{bmatrix}$ has at most $\min(n, p)$ pivot columns, because each column contains at most one pivot position, and each row contains at most one pivot position. Therefore if $p > n$ then A does not have a pivot position in every column so its columns are linearly dependent. □

Proposition. Suppose $v_1, v_2, \dots, v_p \in \mathbb{R}^n$. If some $v_i = 0$ then the list of vectors is linearly dependent.

Proof. If $v_i = 0$ then the matrix $\begin{bmatrix} v_1 & v_2 & \dots & v_p \end{bmatrix}$ has no pivot in column i. □

4 Linear transformations

A *function* f takes an input x from some set X and produces an output $f(x)$ in another set Y.

We write $f : X \to Y$ to mean that f is a function that takes inputs from X and gives outputs in Y.

The set X is called the *domain* of the function f. The set Y is called the *codomain* of f.

Every element $x \in X$ is a valid input to f. However, not every $y \in Y$ needs to occur as an output of f.

Definition. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function whose domain and codomain are sets of vectors. The function f is a *linear transformation* (also called a *linear function*) if both of these properties hold:

(1) $f(u+v) = f(u) + f(v)$ for all vectors $u, v \in \mathbb{R}^n$.

(2) $f(cv) = cf(v)$ for all vectors $v \in \mathbb{R}^n$ and scalars $c \in \mathbb{R}$.

Example. If A is an $m \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^m$ is the function with the formula $T(v) = Av$ for $v \in \mathbb{R}^n$ then T is a linear function.

Linear transformations have some additional properties worth noting:

Proposition. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then

(3) $f(0) = 0$.

(4) $f(u-v) = f(u) - f(v)$ for $u, v \in \mathbb{R}^n$.

(5) $f(a_1v_1 + a_2v_2 + \cdots + a_pv_p) = a_1f(v_1) + \cdots + a_pf(v_p)$ for any $a_i \in \mathbb{R}$ and $v_i \in \mathbb{R}^n$.

Proof. We have $2f(0) = f(0+0) = f(0)$ so $f(0) = 0$.

We have $f(u-v) = f(u) + f(-v) = f(u) + (-1)f(v) = f(u) - f(v)$.

Finally, we have $f(a_1v_2 + a_2v_2) = f(a_1v_1) + f(a_2v_2) = a_1f(v_1) + a_2f(v_2)$.

The proves (5) when $p = 2$, and the argument for $p > 2$ is similar (or can be deduced by *induction*). $\square$

Define $e_1, e_2, \dots, e_n \in \mathbb{R}^n$ as the vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad e_{n-1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The following identities are very basic but important observations.

Fact. If $w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{R}^n$ then $w = \begin{bmatrix} w_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ w_2 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ w_n \end{bmatrix} = w_1e_1 + w_2e_2 + \cdots + w_ne_n$.

Fact. If A is an $m \times n$ matrix then Ae_i is the ith column of A.

For example $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} e_3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

Here is the fundamental theorem relating matrices and linear transformations:

Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.

Then there is a unique $m \times n$ matrix A such that $T(v) = Av$ for all $v \in \mathbb{R}^n$.

The matrix A has the exact formula $A = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) & \dots & T(e_n) \end{bmatrix}$.

In this sense **matrices uniquely represent linear transformations $\mathbb{R}^n \to \mathbb{R}^m$.**

Proof. Define $A = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) & \dots & T(e_n) \end{bmatrix}$ as in the statement of the theorem.

This is an $m \times n$ matrix because each vector $T(e_i)$ is in $\mathbb{R}^m$.

If $w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{R}^n$ then $T(w) = T(w_1e_1 + \dots + w_ne_n) = w_1T(e_1) + \dots + w_nT(e_n) = Aw$.

Thus A is one matrix such that $T(v) = Av$ for all vectors $v \in \mathbb{R}^n$.

To show that A is the only such matrix, suppose B is a $m \times n$ matrix with $T(v) = Bv$ for all $v \in \mathbb{R}^n$.

Then $T(e_i) = Ae_i = Be_i$ for all $i = 1, 2, \dots, n$.

But Ae_i and Be_i are the ith columns of A and B.

Therefore A and B have the same columns, so they are the same matrix: $A = B$. $\square$

We call the matrix A in this theorem the *standard matrix* of the linear transformation T.

Example. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is the function $T(v) = 3v$.

This is a linear transformation. What is the standard matrix A of T?

Using the formula in the previous theorem, the standard matrix of $T : \mathbb{R}^n \to \mathbb{R}^n$ is

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{bmatrix} = \begin{bmatrix} 3e_1 & 3e_2 & \dots & 3e_n \end{bmatrix} = \begin{bmatrix} 3 & 0 & \dots & 0 \\ 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 3 \end{bmatrix}.$$

This matrix has nonzero entries only in positions $(1, 1), (2, 2), \dots, (n, n)$. One calls such a matrix *diagonal*.

Example. Suppose $T : \mathbb{R}^n \to \mathbb{R}$ is the function

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1^2 + v_2^2 + \dots + v_n^2.$$

This function is *not* linear: we have $T(2v) = 4T(v) \neq 2T(v)$ for any nonzero vector $v \in \mathbb{R}^n$.

Example. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is the function $T\left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) = \begin{bmatrix} v_n \\ \vdots \\ v_2 \\ v_1 \end{bmatrix}$.

This function is a linear transformation. (Why?) Its standard matrix is

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_{n-1}) & T(e_n) \end{bmatrix} = \begin{bmatrix} e_n & e_{n-1} & \dots & e_2 & e_1 \end{bmatrix} = \begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix}.$$

In the matrix on the right, the blank space indicates positions in the matrix containing zero entries.

5 Vocabulary

Keywords from today's lecture:

1. **Linearly independent** vectors.

   Vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ are **linearly independent** if $x_1v_1 + \cdots + x_pv_p = 0$ holds only if $x_1 = x_2 = \cdots = x_p = 0$; or when $\begin{bmatrix} v_1 & v_2 & \dots & v_p \end{bmatrix}$ has a pivot position in every column.

   Vectors that are not linearly independent are **linearly dependent**.

   Example: The three vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$ are linearly independent.

   The four vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$ are linearly dependent.

2. **Domain** and **codomain** of a function $f : X \to Y$.

   The **domain** X is the set of inputs for the function.

   The **codomain** Y is a set that contains the output of the function. This set can also contain elements that are not outputs of the function.

   Example: If A is an $m \times n$ matrix then the function $T(v) = Av$ has domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$.

3. **Linear function** $f : \mathbb{R}^n \to \mathbb{R}^m$.

   A function with $f(cv) = cf(v)$ and $f(u + v) = f(u) + f(v)$ for $c \in \mathbb{R}$ and $u, v \in \mathbb{R}^n$.

   Example: Every such function has the form $f(v) = Av$ for a unique $m \times n$ matrix A.

   The matrix A is called the **standard matrix** of f if $f(v) = Av$ for all $v \in \mathbb{R}^n$.

4. **Diagonal** matrix

   A matrix which has 0 in position (i, j) if $i \neq j$.

   Example: $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$.