

This document is an **exact transcript** of the lecture, with extra summary and vocabulary sections. Due to time constraints, the lectures sometimes only contain limited illustrations, proofs, and examples. For a more thorough discussion of the course content, **consult the textbook**.

Summary

Quick summary of today's notes. Lecture starts on next page.

- Let H be a subspace of $\mathbb{R}^n$.

Every basis of H has the same size.

The size of any basis of H is called its *dimension*. This number is denoted $\dim H$.

We always have $0 \leq \dim H \leq n$.

If $\dim H = d$ then we say that H is *d-dimensional*.

Dimension measures the size of a subspace.

We usually do not think of individual vectors as having dimension, since a single vector belongs to many different subspaces at the same time, all with different dimensions.

- Only the zero subspace has dimension 0.

The only subspace of $\mathbb{R}^n$ with dimension n is $\mathbb{R}^n$ itself.

If $U \subseteq V \subseteq \mathbb{R}^n$ are subspaces then $0 \leq \dim U \leq \dim V \leq n$.

- If $\mathcal{B} = (v_1, v_2, \dots, v_m)$ is an ordered basis for a subspace H of $\mathbb{R}^n$, then each $h \in H$ has the form

$$h = \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} \quad \text{for a unique vector} \quad \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} \in \mathbb{R}^m.$$

The vector on the right is the *coordinate vector* of h in the basis $\mathcal{B}$, sometimes denoted $[h]_{\mathcal{B}} \in \mathbb{R}^m$.

- Let A be an $m \times n$ matrix.

The dimension of $\text{Col } A$ is the number of pivot columns in A .

The dimension of $\text{Nul } A$ is the number of non-pivot columns in A .

Consequently $\dim \text{Col } A + \dim \text{Nul } A = n =$ the total number of columns in A .

- The *rank* of A is defined to be $\text{rank } A = \dim \text{Col } A$.

A is invertible if and only if $\text{rank } A = m = n$.

Assume $m = n$. Then A is invertible if and only if $\text{Nul } A = \{0\}$.

- Suppose H of $\mathbb{R}^n$ is a subspace and $p = \dim H$.

Any set of p linearly independent vectors in H is a basis for H .

Any set of p vectors whose span in H is a basis for H .

1 Last time: subspaces

Definition. A subset H of $\mathbb{R}^n$ is a *subspace* if $0 \in H$, $u + v \in H$, and $cv \in H$ for all $u, v \in H$ and $c \in \mathbb{R}$. Equivalently, a subspace is a nonempty set that contains all linear combinations of vectors in the set.

Example. Examples of subspaces of $\mathbb{R}^n$:

- The set $\{0\}$ containing just the zero vector.
- The set $\mathbb{R}^n$ itself.
- The set of all scalar multiples of a single vector.
- The span of any set of vectors in $\mathbb{R}^n$.
- The range of a linear function $T : \mathbb{R}^k \rightarrow \mathbb{R}^n$.
- The set of vectors v with $T(v) = 0$ for a linear function $T : \mathbb{R}^n \rightarrow \mathbb{R}^k$.

The union of two subspaces is not necessarily a subspace. (Consider two lines in $\mathbb{R}^2$)

The sum of two subspaces U and V is the set $U + V = \{u + v : u \in U \text{ and } v \in V\}$. This is a subspace.

The intersection of two subspaces is always a subspace. (Check the conditions defining a subspace.)

Definition. To any $m \times n$ matrix A there are two corresponding subspaces of interest:

1. The *column space* of A is the subspace $\text{Col } A = \{Ax : x \in \mathbb{R}^n\}$ of $\mathbb{R}^m$.
2. The *null space* of A is the subspace $\text{Nul } A = \{x \in \mathbb{R}^n : Ax = 0\}$ of $\mathbb{R}^n$.

It is not obvious from these definitions, but it will turn out that each subspace of $\mathbb{R}^m$ occurs as the column space of some matrix. Likewise, each subspace of $\mathbb{R}^n$ occurs as the null space of some matrix.

If A and B are matrices with the same number of rows then $\text{Col} \begin{bmatrix} A & B \end{bmatrix} = \text{Col } A + \text{Col } B$.

If A and B are matrices with the same number of columns then $\text{Nul} \begin{bmatrix} A \\ B \end{bmatrix} = \text{Nul } A \cap \text{Nul } B$.

Definition. A *basis* of a subspace H of $\mathbb{R}^n$ is a set of linearly independent vectors whose span is H .

An important basis with its own notation: the *standard basis* of $\mathbb{R}^n$ consists of the vectors $e_1, e_2, \dots, e_n$ where e_i is the vector in $\mathbb{R}^n$ with 1 in row i and 0 in all other rows.

Theorem. Every subspace H of $\mathbb{R}^n$ has a basis of size at most n .

Let A be an $m \times n$ matrix.

How to find a basis of $\text{Col } A$.

1. Compute $\text{RREF}(A)$ and circle the first nonzero entry in each nonzero row (going left to right).
2. Select the columns of A containing the circled positions. These *pivot columns* are a basis for $\text{Col } A$.

How to find a basis of $\text{Nul } A$.

1. Find all solutions to $Ax = 0$ by row reducing A to echelon form. Recall that x_i is a *basic variable* if column i of $\text{RREF}(A)$ contains a leading 1, and that otherwise x_i is a *free variable*.

2. Express each basic variable in terms of the free variables, and then write

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_{i_1}b_1 + x_{i_2}b_2 + \cdots + x_{i_k}b_k$$

where $x_{i_1}, x_{i_2}, \dots, x_{i_k}$ are the free variables and $b_1, b_2, \dots, b_k \in \mathbb{R}^n$.

3. The vectors $b_1, b_2, \dots, b_k$ then form a basis for $\text{Nul } A$.

Example. Suppose $A = \begin{bmatrix} 1 & 2 & 5 & 8 \\ 2 & 3 & 7 & 0 \end{bmatrix}$.

- Then $A \sim \begin{bmatrix} 1 & 2 & 5 & 8 \\ 0 & -1 & -3 & -16 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -24 \\ 0 & 1 & 3 & 16 \end{bmatrix} = \text{RREF}(A)$.
- Therefore the set of vectors $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$ is a basis for $\text{Col } A$.
- We can also conclude from the computation of $\text{RREF}(A)$ that $Ax = 0$ if and only if

$$\begin{cases} x_1 - x_3 - 24x_4 = 0 \\ x_2 + 3x_3 + 16x_4 = 0 \end{cases} \quad \text{which means} \quad \begin{cases} x_1 = x_3 + 24x_4 \\ x_2 = -3x_3 - 16x_4 \end{cases}$$

Thus x_1, x_2 are basic variables, x_3, x_4 are free variables, and if $Ax = 0$ then

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 + 24x_4 \\ -3x_3 - 16x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 24 \\ -16 \\ 0 \\ 1 \end{bmatrix}.$$

- The set of vectors $\left\{ \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 24 \\ -16 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\text{Nul } A$.

2 Coordinate systems

Suppose H is a subspace of $\mathbb{R}^n$. Let $b_1, b_2, \dots, b_k$ be a basis of H .

Theorem. Let $v \in H$. There are unique coefficients $c_1, c_2, \dots, c_k \in \mathbb{R}$ such that

$$c_1b_1 + c_2b_2 + \cdots + c_kb_k = v.$$

Proof. Consider the matrix $B = [b_1 \ b_2 \ \cdots \ b_k]$. Suppose $x \in \mathbb{R}^k$ is a vector of variables.

The matrix equation $Bx = v$ has at least one solution since the columns of B span H .

The matrix equation $Bx = v$ has at most one solution because the columns of B are linearly independent.

(This holds as if $x \neq y$ and $Bx = By = v$ then $x - y$ would be a nonzero vector with $B(x - y) = 0$.)

The coefficients $c_1, c_2, \dots, c_k$ are just the entries in the unique solution to $Bx = v$. $\square$

Let $\mathcal{B} = (b_1, b_2, \dots, b_k)$ be the list containing our basis vectors in some fixed order.

Given $v \in H$, define $[v]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} \in \mathbb{R}^k$ as the unique vector with $c_1 b_1 + c_2 b_2 + \dots + c_k b_k = v$.

As mentioned in the proof, $[v]_{\mathcal{B}}$ is the unique solution to the matrix equation $\begin{bmatrix} b_1 & b_2 & \dots & b_k \end{bmatrix} x = v$.

We call $[v]_{\mathcal{B}}$ the *coordinate vector of v in the basis $\mathcal{B}$* or just *v in the basis $\mathcal{B}$* .

Example. If $H = \mathbb{R}^n$ and $\mathcal{B} = (e_1, e_2, \dots, e_n)$ is the standard basis then $[v]_{\mathcal{B}} = v$.

Example. If $H = \mathbb{R}^n$ and $\mathcal{B} = (e_n, \dots, e_2, e_1)$ and $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ then $[v]_{\mathcal{B}} = \begin{bmatrix} v_n \\ \vdots \\ v_2 \\ v_1 \end{bmatrix}$.

Example. Let $b_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$ and $b_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$.

Then $\mathcal{B} = (b_1, b_2)$ is a basis for $H = \mathbb{R}\text{-span}\{b_1, b_2\}$, which is a subspace of $\mathbb{R}^3$.

The unique $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ such that $\begin{bmatrix} 3 & -1 \\ 6 & 0 \\ 2 & 1 \end{bmatrix} x = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$ is found by row reduction:

$$\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 3 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -3 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

The last matrix implies that $x_1 = 2$ and $x_2 = 3$ so $[v]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Example. If $b_1 = e_1 - e_2$, $b_2 = e_2 - e_3$, $b_3 = e_3 - e_4$, $\dots$, $b_{n-1} = e_{n-1} - e_n$ and

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ -v_1 - v_2 - \dots - v_{n-1} \end{bmatrix}$$

then $v \in H = \mathbb{R}\text{-span}\{b_1, b_2, \dots, b_{n-1}\}$ and

$$[v]_{\mathcal{B}} = \begin{bmatrix} v_1 \\ v_1 + v_2 \\ v_1 + v_2 + v_3 \\ v_1 + v_2 + v_3 + v_4 \\ \vdots \\ v_1 + v_2 + v_3 + \dots + v_{n-1} \end{bmatrix} \in \mathbb{R}^{n-1}.$$

The notation $[v]_{\mathcal{B}}$ gives us an easy way to check the following important property:

Theorem. Let H be a subspace of $\mathbb{R}^n$. Then all bases of H have the same number of elements.

Proof. Suppose $\mathcal{B} = (b_1, b_2, \dots, b_k)$ and $\mathcal{B}' = (b'_1, b'_2, \dots, b'_l)$ are two (ordered) bases of H with $k < l$.

Then $[b'_1]_{\mathcal{B}}, [b'_2]_{\mathcal{B}}, \dots, [b'_l]_{\mathcal{B}}$ are $l > k$ vectors in $\mathbb{R}^k$, so they must be linearly dependent.

This means there exist coefficients $c_1, c_2, \dots, c_l \in \mathbb{R}$, not all zero, with

$$c_1[b'_1]_{\mathcal{B}} + c_2[b'_2]_{\mathcal{B}} + \dots + c_l[b'_l]_{\mathcal{B}} = 0.$$

But we have $c_1[b'_1]_{\mathcal{B}} + c_2[b'_2]_{\mathcal{B}} + \dots + c_l[b'_l]_{\mathcal{B}} = [c_1b'_1 + c_2b'_2 + \dots + c_lb'_l]_{\mathcal{B}}$.

(This is the key step; why is this true? Think about how $[v]_{\mathcal{B}}$ is defined.)

Thus $[c_1b'_1 + c_2b'_2 + \dots + c_lb'_l]_{\mathcal{B}} = 0$, so

$$c_1b'_1 + c_2b'_2 + \dots + c_lb'_l = \begin{bmatrix} b_1 & b_2 & \dots & b_k \end{bmatrix} [c_1b'_1 + c_2b'_2 + \dots + c_lb'_l]_{\mathcal{B}} = 0.$$

(The first equality holds since by definition $v = \begin{bmatrix} b_1 & b_2 & \dots & b_k \end{bmatrix} [v]_{\mathcal{B}}$.)

Since the coefficients c_i are not all zero, this contradicts the fact that $b'_1, b'_2, \dots, b'_l$ are linearly independent. This means our original assumption that H has two bases of different sizes is impossible. $\square$

3 Dimension

Let $\mathcal{B} = (b_1, b_2, \dots, b_k)$ be an ordered basis of a subspace H of $\mathbb{R}^n$.

The function $H \rightarrow \mathbb{R}^k$ with the formula $v \mapsto [v]_{\mathcal{B}}$ is linear and invertible.

Thus H “looks the same as” $\mathbb{R}^k$. For this reason we say that H is *k -dimensional*. More generally:

Definition. The *dimension* of a subspace H is the number of vectors in any basis of H .

We denote the dimension of H by $\dim H$. This number belongs to $\{0, 1, 2, 3, \dots\}$.

The only way we can have $\dim H = 0$ is if $H = \{0\}$ is the zero subspace.

Example. We have $\dim \mathbb{R}^n = n$ since the standard basis of $\mathbb{R}^n$ has n elements.

If H is the set of all vectors of the form $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$, then H is a subspace and $\dim H = k$.

Note that $\{e_1, e_2, \dots, e_k\} \subset \mathbb{R}^n$ is a basis for H , but the subspace H is **not equal** to $\mathbb{R}^k$.

A line in $\mathbb{R}^2$ through the origin is a 1-dimensional subspace.

Let A be an $m \times n$ matrix. The processes we gave to construct bases of $\text{Nul } A$ and $\text{Col } A$ imply that:

Corollary. The dimension of $\text{Nul } A$ is the number of free variables in the linear system $Ax = 0$.

Corollary. The dimension of $\text{Col } A$ is the number of pivot columns in A .

There is a special name for the dimension of the column space of a matrix:

Definition. The *rank* of a matrix A is $\text{rank } A = \dim \text{Col } A$.

Putting everything together gives the following pair of important results.

Theorem (*Rank-nullity theorem*). If A is a matrix with n columns then $\text{rank } A + \dim \text{Nul } A = n$.

Proof. The number of free variables in the system $Ax = 0$ is also the number non-pivot columns in A .

Therefore $\text{rank } A + \dim \text{Nul } A$ is the total number of columns in A . $\square$

Theorem (*Basis theorem*). If H is a subspace of $\mathbb{R}^n$ with $\dim H = p$ then

1. Any set of p linearly independent vectors in H is a basis for H .
2. Any set of p vectors in H whose span is H is a basis for H .

Proof. We prove the two parts separately.

1. Suppose we have p linearly independent vectors in H .

If these vectors do not span H , then adding a vector that is in H but not in their span gives a set of $p + 1$ linearly independent vectors in H .

If this larger set still does not span H , then adding a vector from H that is not in the span gives an even larger linearly independent set of $p + 2$ vectors.

Continuing in this way must eventually produce a basis for H , but this basis will have more than p elements, contradicting $\dim H = p$.

2. Suppose we instead have p vectors whose span is H .

If these vectors are linearly dependent, then one of the vectors is a linear combination of the others. Remove this vector to get $p - 1$ vectors that span H .

If these vectors are still not linearly independent, then one is a linear combination of the others and removing this vector gives a set of $p - 2$ vectors that span H .

Continuing in this way must eventually produce a basis for H , but this basis will have fewer than p elements, contradicting $\dim H = p$. $\square$

Corollary. If H is an n -dimensional subspace of $\mathbb{R}^n$ then $H = \mathbb{R}^n$.

Proof. If H has a basis with n elements then these elements are linearly independent, so form a basis for $\mathbb{R}^n$. Then every vector in $\mathbb{R}^n$ is a linear combination of the basis vectors, so belongs to H . $\square$

Recall that if U and V are two sets then we write “ $U \subset V$ ” or “ $U \subseteq V$ ” to mean that every element of U is also an element of V . Both notations mean the same thing. If $U \subseteq V$ then we could have $U = V$.

On the other hand, writing “ $U \subsetneq V$ ” means “ $U \subseteq V$ but $U \neq V$.”

It holds that $U = V$ if and only if we have both $U \subseteq V$ and $V \subseteq U$.

Corollary. If $U, V \subseteq \mathbb{R}^n$ are subspaces with $U \subsetneq V$, then $\dim U < \dim V \leq n$.

Proof. If $j = \dim V \leq \dim U = k$ and $u_1, u_2, \dots, u_k$ is a basis for U , then $u_1, u_2, \dots, u_j$ would be linearly independent and therefore a basis for V . But then $V \subseteq U$ which would imply $U = V$ if also $U \subseteq V$. $\square$

Corollary. Let A be an $n \times n$ matrix. The following are equivalent:

- (a) A is invertible.
- (b) The columns of A form a basis for $\mathbb{R}^n$.
- (c) $\text{rank } A = \dim \text{Col } A = n$.
- (d) $\dim \text{Nul } A = 0$.

Proof. We have already seen that (a) and (b) are equivalent.

(c) holds if and only if the columns of A span $\mathbb{R}^n$ which is equivalent to (a).

(d) holds if and only if the columns of A are linearly independent which is equivalent to (a). $\square$

4 Vocabulary

Keywords from today's lecture:

1. **Coordinate vector** of a vector $v \in H$ with respect to an ordered basis $\mathcal{B} = (b_1, b_2, \dots, b_k)$.

The unique vector of coefficients $[v]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} \in \mathbb{R}^k$ with $c_1 b_1 + c_2 b_2 + \dots + c_k b_k = v$.

Example: If $H = \mathbb{R}^2$ and $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$ and $v = \begin{bmatrix} x \\ y \end{bmatrix}$ then $[v]_{\mathcal{B}} = \begin{bmatrix} x - y \\ y \end{bmatrix}$.

2. **Dimension** of a subspace $H \subseteq \mathbb{R}^n$

The number $\dim H$ of vectors in any basis for H .

3. **Rank** of an $m \times n$ matrix A .

The dimension of the column space $\text{Col } A$. This is also the number of pivot columns in A .

This is denoted $\text{rank } A$.

4. **Rank-nullity theorem.**

If A is an $m \times n$ matrix then $\dim \text{Col } A + \dim \text{Nul } A = \text{rank } A + \dim \text{Nul } A = n$.

5. **Basis theorem.**

If $H \subseteq \mathbb{R}^n$ is a subspace with $\dim H = p$ then (1) any set of p linearly independent vectors in H is a basis for H and (2) any set of p vectors whose span is H is a basis for H .