Instructions: Choose **4 problems** and **write down detailed solutions**.

Some of the problems are more challenging than others. Harder problems are marked with a star.

You can use any resources to solve the problems, but **you are encouraged to attempt to do the problems yourself** to best prepare for our examinations.

The standard for full credit is low: you just need to make a good-faith attempt on 4 problems.

- You can receive full credit even if your final answer is incorrect as long as you show all of your work.
- Conversely, if you just give an answer without explaining how it was derived, you may lose points.

Submission: Please handwrite your answers and show all steps in your calculations, as you would on an exam. **Submit your hard copy solutions** before the end of the day on the due date to your tutorial's homework submission box outside the 3rd floor MATH Admin Offices near Lift 25/26.

Please **coordinate with your tutorial TA directly** if you need to submit solutions electronically.

1. Find general formulas for the solutions of the linear systems

$$\begin{cases} x_1 + 2x_2 = 1 \\ 3x_1 + 4x_2 = 3 \end{cases} \quad \text{and} \quad \begin{cases} x_1 + 2x_2 = 1 \\ 3x_1 + 6x_2 = 3. \end{cases}$$

   Now suppose a, b, c, d, p, q are real numbers with $ad - bc \neq 0$.

   Find a general formula for the solution of the linear system $\begin{cases} ax_1 + bx_2 = p \\ cx_1 + dx_2 = q. \end{cases}$

2. Suppose $a_1, a_2, \dots, a_8 \in \mathbb{R}$ are numbers and x_1, x_2, x_3, x_4 are variables arranged in a grid:

$$\begin{array}{cccc} & a_1 & a_2 & \\ a_3 & x_1 & x_2 & a_4 \\ a_5 & x_3 & x_4 & a_6 \\ & a_7 & a_8 & \end{array}$$

   Each interior point in the grid is the average of its 4 neighbors. For example,

$$x_1 = \frac{a_1 + a_3 + x_2 + x_3}{4}.$$

   These conditions mean that we can interpret x_1, x_2, x_3, x_4 as interior temperature readings on a flat surface whose boundary temperatures are the numbers $a_1, a_2, \dots, a_8$.

   Write down the linear system that corresponds to this setup.

   Determine how many solutions exist to this system, and find a formula for all solutions.

3. (*) Suppose n is a positive integer and $a_1, a_2, \dots, a_n$ are real numbers.

   What condition must be satisfied for the linear system

$$\begin{cases} x_1 - x_2 = a_1 \\ x_2 - x_3 = a_2 \\ x_3 - x_4 = a_3 \\ \vdots \\ x_{n-1} - x_n = a_{n-1} \\ x_n - x_1 = a_n \end{cases}$$

   to be consistent? Write down the augmented matrix of this linear system. Assuming the system is consistent, find a general formula for its solutions.

4. Suppose we have two linear systems with the same number of equations and the same number of variables. Then the systems' augmented matrices have the same size. If the augmented matrices are row equivalent then the systems are equivalent, meaning they have the same solutions. Do there exist two linear systems, both with m equations and n variables, that are equivalent but whose augmented matrices are **not** row equivalent? Explain why this is impossible or find an example.

5. There is a way to multiply two 2×2 matrices to get another 2×2 matrix. The formula is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{bmatrix}.$$

Find 2×2 matrices D, E, and F with all real entries such that

$$D \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix}, \quad E \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}, \quad \text{and} \quad F \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+3c & b+3d \\ c & d \end{bmatrix}$$

for all $a, b, c, d \in \mathbb{R}$. Once you find your matrices, compute

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} D, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} E, \quad \text{and} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} F.$$

6. Find all matrices $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ with $x_1, x_2, x_3, x_4 \in \mathbb{R}$ such that

$$X^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

7. (*) The matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has the property that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

for all $a, b, c, d \in \mathbb{R}$, while the matrix $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has the property that

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Assume $a, b, c \in \mathbb{R}$ and $a \neq 0$. The quadratic formula says that the solutions to

$$ax^2 + bx + c = 0$$

are $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$, which are not real numbers if $b^2 - 4ac < 0$.

Can you always find a matrix $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ with all real entries $x_1, x_2, x_3, x_4 \in \mathbb{R}$ that solves

$$aX^2 + bX + cI = \mathbf{0}?$$

If this is possible, find a formula for such a solution in terms of a, b, and c.

(Here we multiply numbers and matrices like this: $bX = \begin{bmatrix} bx_1 & bx_2 \\ bx_3 & bx_4 \end{bmatrix}$.)

8. (*) Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2×2 matrix with real entries such that $ad - bc \neq 0 \neq b$.

Show that there are numbers $p_1, p_2, p_3, q_1, q_2, q_3 \in \mathbb{R}$ with

$$A = \begin{bmatrix} p_1 & 0 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 & 0 \\ q_2 & q_3 \end{bmatrix}.$$

9. If A is a 1×3 matrix then $\mathsf{RREF}(A)$ either has the form

$$\begin{bmatrix} 1 & * & * \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 & * \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

where each $*$ means an arbitrary real number. In the first, second, and fourth cases, A is the augmented matrix of a linear system in two variables with infinitely many solutions. In the third case, A is the augmented matrix of a linear system in two variables with zero solutions.

Describe with similar notation what the possibilities are for $\mathsf{RREF}(A)$ if A is a 2×3 matrix. In each case, indicate how many solutions there are for the linear system whose augmented matrix is A.

10. What are the possibilities for $\mathsf{RREF}\left(\begin{bmatrix} x & 1 \\ y & 2 \end{bmatrix}\right)$ if x and y are arbitrary real numbers? Draw a picture of the xy-plane in which you identify the regions of points (x, y) where $\mathsf{RREF}\left(\begin{bmatrix} x & 1 \\ y & 2 \end{bmatrix}\right)$ takes its different possible values.

11. If the reduced echelon form of the augmented matrix of some linear system is

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 5 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

then what is the general formula for the solution to the linear system?

Going in the opposite direction, find a matrix A that is the augmented matrix of a linear system with 3 equations and 4 variables whose general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 - 3a - 4b \\ a \\ 2 - 5b \\ b \end{bmatrix} \text{ for all } a, b \in \mathbb{R}.$$

12. Arrange the first 9 digits of your student ID as the entries of a 3×3 matrix A. (If you run out of digits, reuse the first few digits.) Compute $\mathsf{RREF}(A)$ by hand without using a calculator, showing all of your work in the intermediate steps of the row reduction algorithm.

13. (*) A *linear inequality* is an equation of the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \geq b$$

where $a_1, a_2, \ldots, a_m, b \in \mathbb{R}$ are numbers and $x_1, x_2, \ldots, x_n$ are variables. A solution to a linear inequality is an assignment of numbers to the variables which makes the inequality true. Systems of linear inequalities and their solutions are defined similarly.

Make a picture of the lines $x_1 + x_2 = 0$ and $x_1 - x_2 = 4$ and $2x_1 + x_2 = 6$ and compute the three points where two of the lines intersect. These lines divide the $\mathbb{R}^2$ plane into 7 regions.

Explain why each of these regions is the set of solutions to a system of linear inequalities in the variables x_1 and x_2. Identify the corresponding system for each region.

14. (*) Consider the set $\mathcal{X}$ of 3×3 matrices A whose entries are each equal to $+1$ or -1.

There are $2^9 = 512$ matrices in this set.

What matrices E can occur as $\mathsf{RREF}(A)$ if $A \in \mathcal{X}$?

For each such matrix E, find the **exact number** of matrices $A \in \mathcal{X}$ with $\mathsf{RREF}(A) = E$.

15. Define M to be the $n \times n$ matrix whose entry in position (i, j) is $M_{ij} = i + j$ for all $i, j = 1, 2, \ldots, n$.

Find an exact formula for $\mathsf{RREF}(M)$.