Instructions: Choose **4 problems** and **write down detailed solutions**.

Some of the problems are more challenging than others. Harder problems are marked with a star.

You can use any resources to solve the problems, but **you are encouraged to attempt to do the problems yourself** to best prepare for our examinations.

The standard for full credit is low: you just need to make a good-faith attempt on 4 problems.

- You can receive full credit even if your final answer is incorrect as long as you show all of your work.
- Conversely, if you just give an answer without explaining how it was derived, you may lose points.

Submission: Please handwrite your answers and show all steps in your calculations, as you would on an exam. **Submit your hard copy solutions** before the end of the day on the due date to your tutorial's homework submission box outside the 3rd floor MATH Admin Offices near Lift 25/26.

Please **coordinate with your tutorial TA directly** if you need to submit solutions electronically.

1. Assume h and k are real numbers and consider the linear system

$$\begin{aligned} x_1 + 4x_2 &= k \\ 3x_1 + hx_2 &= 7. \end{aligned}$$

Determine all values of h and k such that this system has (i) zero solutions, (ii) a unique solution, or (iii) infinitely many solutions.

2. Construct a visual "proof" that if $v \in \mathbb{R}^2$ and $w \in \mathbb{R}^2$ are two nonzero vectors that are not on the same line, then $\mathbb{R}\text{-span}\{v, w\} = \mathbb{R}^2$. Do this in the following way.

(a) First, draw a picture of the xy-plane with two arrows that represent an arbitrary choice of nonzero vectors $v, w \in \mathbb{R}^2$ that are not on the same line.

You don't need to assign any specific numerical values to the coordinates of these vectors.

(b) Then pick a random point (not equal to the endpoint of either of your vectors) from each of the four quadrants of the xy-plane. For each point, draw a picture containing three vectors: the arrow that goes from the origin to your chosen point, along with two arrows that are scalar multiples of v and w and which add up to the first arrow.

Your final answer should consist of these 5 pictures. These pictures should convince you that if $u \in \mathbb{R}^2$ is any vector then we can find scalars $a, b \in \mathbb{R}$ such that $u = av + bw$.

3. Suppose $v_1 = \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}$, $v_2 = \begin{bmatrix} -3 \\ 2 \\ 7 \end{bmatrix}$, $v_3 = \begin{bmatrix} 3 \\ -6 \\ -9 \end{bmatrix}$, and $w = \begin{bmatrix} 6 \\ -10 \\ a \end{bmatrix}$ where $a \in \mathbb{R}$.

Find all values of a such that w is in $\mathbb{R}\text{-span}\{v_1, v_2, v_3\}$.

4. Describe all matrices A such that $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$ and $A \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$.

5. Assume A is a 3×3 matrix. Only the first and second columns of A are pivot columns and

$$A \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

What is $\mathsf{RREF}(A)$? What is the general solution to $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$?

6. Find all values of $h \in \mathbb{R}$ for which the vectors

$$\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} -5 \\ 5 \\ 11 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 7 \\ h \end{bmatrix}$$

are linearly dependent.

7. Suppose $v_1, v_2, \dots, v_k \in \mathbb{R}^n$ are linearly independent vectors.
   (a) If we add another vector v_{k+1} to this list, decide whether the larger list is ALWAYS, SOMETIMES, or NEVER linearly independent.
   (b) If we delete one of the vectors from the list, say v_k, decide whether the smaller list is ALWAYS, SOMETIMES, or NEVER linearly independent.

   For each part, if your answer is ALWAYS or NEVER, then explain why this is justified. If your answer is SOMETIMES, then provide examples showing that both outcomes are possible.

8. Suppose $c_1, c_2, c_3 \in \mathbb{R}$ are some real numbers.

   Let $V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3 : c_1 a_1 + c_2 a_2 + c_3 a_3 = 0 \right\}$.

   Find a set of vectors that span V.

   Your answer should have 8 cases depending on which of c_1, c_2, and c_3 are equal to zero.

9. Let $V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n : a_1 + a_2 + \dots + a_n = 0 \right\}$.

   Find a set of $n-1$ vectors $v_1, v_2, \dots, v_{n-1}$ in $\mathbb{R}^n$ with span equal to V.

10. (*) Choose an angle $\theta \in [0, 2\pi)$.

    What is the relationship between the vector $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$ and $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$?

    Draw a picture and justify your answer.

    Then use your answer to prove the trigonometric sum-angle formulas

$$\cos(\theta + \phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi) \quad \text{and} \quad \sin(\theta + \phi) = \cos(\theta)\sin(\phi) + \sin(\theta)\cos(\phi).$$

    Hint: answer the question for $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ v_2 \end{bmatrix}$*, using the unit circle definitions of* sin *and* cos*. Then use parallelogram rule for vector addition to deduce the general answer.*

11. Suppose a and b are real numbers. Consider the lines

$$L_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : x_2 = a x_1 \right\} \quad \text{and} \quad L_2 = \left\{ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2 : y_2 = b y_1 \right\}.$$

    Find all values of a and b such that there is exactly one way of writing

$$\begin{bmatrix} 2026 \\ 2121 \end{bmatrix} = v + w$$

    with $v \in L_1$ and $w \in L_2$. Find a formula for v and w in this case.

12. (*) Suppose $v, w \in \mathbb{R}^n$ are vectors with $v_i w_j - v_j w_i = 0$ for all $1 \leq i < j \leq n$. Show that v and w are linearly dependent, that is, one vector is a scalar multiple of the other.

13. Consider the linear system

$$\begin{cases} x_1 + x_2 + x_3 & = 3 \\ 2x_1 - x_2 + x_3 & = 0 \\ 3x_1 + 2x_3 & = 3. \end{cases}$$

Find all solutions (x_1, x_2, x_3) to this system such that **exactly one** of x_1, x_2, or x_3 is an **integer**.

14. Explain how to interpret the statement

"if $S \subset \mathbb{R}^n$ is any set of vectors then $\mathbb{R}\text{-span}(\mathbb{R}\text{-span}\, S) = \mathbb{R}\text{-span}\, S$"

and write down a formal argument that demonstrates why it is true.

15. (*) There are $n \geq 3$ different particles $p_1, p_2, \ldots, p_n$ arranged in a circle. If $1 < i < n$ then p_i is between particles p_{i-1} and p_{i+1}, while p_1 is between p_n and p_2 (and p_n is between p_{n-1} and p_1). The temperature of each particle is the average of the temperatures of its two adjacent neighbors. Must every particle have the same temperature? Justify your answer.