

Instructions: Choose **4 problems** and **write down detailed solutions**.

Some of the problems are more challenging than others. Harder problems are marked with a star.

You can use any resources to solve the problems, but **you are encouraged to attempt to do the problems yourself** to best prepare for our examinations.

The standard for full credit is low: you just need to make a good-faith attempt on 4 problems.

- You can receive full credit even if your final answer is incorrect as long as you show all of your work.
- Conversely, if you just give an answer without explaining how it was derived, you may lose points.

Submission: Please handwrite your answers and show all steps in your calculations, as you would on an exam. **Submit your hard copy solutions** before the end of the day on the due date to your tutorial's homework submission box outside the 3rd floor MATH Admin Offices near Lift 25/26.

Please **coordinate with your tutorial TA directly** if you need to submit solutions electronically.

1. (a) For each $i = 1, 2, 3$, the map $T_i : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ given by the formula

$$T_i(v) = (\text{the vector } v \text{ with row } i \text{ deleted})$$

is a linear transformation. What are the standard matrices of T_1 , T_2 , and T_3 ?

- (b) Can you find two linearly independent vectors $v, w \in \mathbb{R}^3$ such that the three sets

$$\{T_1(v), T_1(w)\}, \quad \{T_2(v), T_2(w)\}, \quad \text{and} \quad \{T_3(v), T_3(w)\}$$

are each linearly dependent? Justify your answer.

2. If $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is a nonzero vector, then $v^\top = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$ and $r_v : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the linear map

$$r_v(x) = x - \frac{v^\top x}{v^\top v} 2v.$$

Write down a formula for the standard matrix A_v of r_v .

Then show that if $u, v \in \mathbb{R}^2$ are both nonzero then $A_u A_v$ is a rotation matrix.

3. Suppose A is an $m \times n$ matrix and B is an $m \times q$ matrix.

The **rank** of a matrix M , denote $\text{rank}(M)$, is its number of pivot columns.

- (a) What relationship, if any, exists between $\text{RREF}(A)$ and $\text{RREF}(\begin{bmatrix} A & B \end{bmatrix})$ in general?
What relationship, if any, exists between $\text{rank}(A)$ and $\text{rank}(\begin{bmatrix} A & B \end{bmatrix})$ in general?
- (b) What relationship, if any, exists between $\text{RREF}(B)$ and $\text{RREF}(\begin{bmatrix} A & B \end{bmatrix})$ in general?
What relationship, if any, exists between $\text{rank}(B)$ and $\text{rank}(\begin{bmatrix} A & B \end{bmatrix})$ in general?
- (c) What relationship, if any, exists between $\text{RREF}(\begin{bmatrix} A & B \end{bmatrix})$ and $\text{RREF}(\begin{bmatrix} B & A \end{bmatrix})$ in general?
What relationship, if any, exists between $\text{rank}(\begin{bmatrix} A & B \end{bmatrix})$ and $\text{rank}(\begin{bmatrix} B & A \end{bmatrix})$ in general?

For the first question in each part, the possible answers are

“the two RREFs are equal” or
“the smaller RREF is a specific submatrix of the larger RREF” or
“no general relationship”.

For the second question in each part, the possible answers are: $\leq$, $\geq$, $=$, or no general relationship.

Pick from among these options, and provide some explanation for your choices.

4. Suppose A is an $m \times n$ matrix and C is a $q \times n$ matrix.

(a) What relationship, if any, exists between $\text{RREF}(A)$ and $\text{RREF}\left(\begin{bmatrix} A \\ C \end{bmatrix}\right)$ in general?

What relationship, if any, exists between $\text{rank}(A)$ and $\text{rank}\left(\begin{bmatrix} A \\ C \end{bmatrix}\right)$ in general?

(b) What relationship, if any, exists between $\text{RREF}(C)$ and $\text{RREF}\left(\begin{bmatrix} A \\ C \end{bmatrix}\right)$ in general?

What relationship, if any, exists between $\text{rank}(C)$ and $\text{rank}\left(\begin{bmatrix} A \\ C \end{bmatrix}\right)$ in general?

(c) What relationship, if any, exists between $\text{RREF}\left(\begin{bmatrix} A \\ C \end{bmatrix}\right)$ and $\text{RREF}\left(\begin{bmatrix} C \\ A \end{bmatrix}\right)$ in general?

What relationship, if any, exists between $\text{rank}\left(\begin{bmatrix} A \\ C \end{bmatrix}\right)$ and $\text{rank}\left(\begin{bmatrix} C \\ A \end{bmatrix}\right)$ in general?

Use the same set of possible answers as in the previous question.

5. Suppose $v_1, v_2, \dots, v_k \in \mathbb{R}^n$ are vectors and $V = \mathbb{R}\text{-span}\{v_1, v_2, \dots, v_k\}$.

(a) If we add another vector v_{k+1} to this list, will it still have the same span?

What can you say about when $V = \mathbb{R}\text{-span}\{v_1, v_2, \dots, v_k, v_{k+1}\}$?

(b) If we delete one of the vectors from the list, say v_k , will it still have the same span?

What can you say about when $V = \mathbb{R}\text{-span}\{v_1, v_2, \dots, v_{k-1}\}$?

Explain and justify your answers to both parts.

6. Let $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ be two vectors in $\mathbb{R}^2$.

The span $\mathbb{R}\text{-span}\{v, w\} = \{\alpha v + \beta w : \alpha, \beta \in \mathbb{R}\}$ is the smallest set of vectors in $\mathbb{R}^2$ containing v and w that is **closed under scalar multiplication and vector addition**.

(a) Explain why $\mathbb{R}\text{-span}\{v, w\} = \mathbb{R}^2$.

(b) Draw a picture of the smallest set $\mathcal{T}$ of vectors in $\mathbb{R}^2$ that contains v and w and is closed under just vector addition. This means that $v \in \mathcal{T}$, $w \in \mathcal{T}$, if $x, y \in \mathcal{T}$ then $x + y \in \mathcal{T}$, and $\mathcal{T}$ is as small as possible with these properties.

(c) Draw a picture of the smallest set $\mathcal{U}$ of vectors in $\mathbb{R}^2$ that contains v and w and is closed under just vector subtraction. This means that $v \in \mathcal{U}$, $w \in \mathcal{U}$, if $x, y \in \mathcal{U}$ then $x - y \in \mathcal{U}$, and $\mathcal{U}$ is as small as possible with these properties.

7. A **constrained linear system** is a linear system with m equations and n variables $x_1, x_2, \dots, x_n$ that imposes the additional constraint that $x_i \geq 0$ for all i .

(a) Give an example of a linear system (for any choice of m and n) with two different solutions whose constrained linear system has only one solution.

(b) If a constrained linear system has two different solutions, then does it have infinitely many? Justify your answer.

8. Suppose $v_1, v_2, \dots, v_k$ and $w_1, w_2, \dots, w_l$ are two lists of vectors in $\mathbb{R}^n$.

Describe an algorithm to determine whether or not $\mathbb{R}\text{-span}\{v_1, v_2, \dots, v_k\} = \mathbb{R}\text{-span}\{w_1, w_2, \dots, w_l\}$.

9. Suppose $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ is a linear transformation.

If $T\left(\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}\right) = T\left(\begin{bmatrix} 4 \\ 4 \\ 7 \end{bmatrix}\right)$ then what is $T\left(\begin{bmatrix} 5 \\ 9 \\ 16 \end{bmatrix}\right)$?

Justify your answer.

10. Suppose $m \leq n$ are positive integers.

Let X be a finite set with m elements and let Y be a finite set with n elements.

- (a) Explain why any one-to-one map $f : Y \rightarrow X$ is also onto.

Similarly, explain why any one-to-one linear map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is also onto.

- (b) Explain why any onto map $f : X \rightarrow Y$ is also one-to-one.

Similarly, explain why any onto linear map $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is also one-to-one.

(Suggestion: use the characterizations of onto and one-to-one linear maps in terms of pivots.)

11. Let $\mathbb{Z}$ be the infinite set of all integers.

The properties in the previous question do not hold for arbitrary maps between infinite sets.

- (a) To show this, give an example of a map $f : \mathbb{Z} \rightarrow \mathbb{Z}$ that is one-to-one but not onto.

- (b) Then give an example of a map $g : \mathbb{Z} \rightarrow \mathbb{Z}$ that is onto but not one-to-one.

12. Consider the set of vectors $\mathcal{Q}$ in $\mathbb{R}^2$ whose endpoints are in the quadrilateral in the xy -plane with vertices $(0, 0)$, $(-20, 23)$, $(21, 21)$, and $(1, 44)$.

If $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation with

$$\left\{ T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) : 0 \leq a \leq 1 \text{ and } 0 \leq b \leq 1 \right\} = \mathcal{Q},$$

then what are the possibilities for the standard matrix A of T ? Justify your answer.

13. (*) The unit cube in $\mathbb{R}^n$ is the set $\mathcal{C} = \{v \in \mathbb{R}^n : 0 \leq v_i \leq 1 \text{ for all } i = 1, 2, \dots, n\}$.

Find all $n \times n$ matrices A such that $A\mathcal{C} = \mathcal{C}$. Be sure to justify your answer.

Here $A\mathcal{C}$ is defined to be the set $\{Av : v \in \mathcal{C}\}$.

14. Suppose $f : X \rightarrow X$ is a function with the same domain and codomain.

For each positive integer n , let $f^n : X \rightarrow X$ be the function with the formula

$$f^n(x) = \underbrace{f(f(\cdots f(x) \cdots))}_{n \text{ times}}$$

so that $f^1(x) = f(x)$, $f^2(x) = f(f(x))$, and $f^3(x) = f(f(f(x)))$, and so on.

The function f has **finite order** if there is a positive integer n with $f^n = \text{id}_X$.

When f has finite order, its **order** is the smallest positive integer n with $f^n = \text{id}_X$.

How many functions $f : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ have finite order?

What are the possible values for their orders? Justify your answer.

15. (*) An $n \times n$ matrix A has **finite order** if there is a positive integer n with $A^n = I$.

When A has finite order, its **order** is the smallest positive integer n with $A^n = I$.

(Notice that if $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the linear function with standard matrix A , then T has finite order if and only if A does, in which case the order of T is the same as the order of A .)

Suppose A is a 2×2 matrix with **integer** entries that has finite order.

What numbers n can occur as the order of A ? Justify your answer.

For each possible value of n , give an example of an integer 2×2 matrix A with order n .