

Instructions: Choose **4 problems** and **write down detailed solutions**.

Some of the problems are more challenging than others. Harder problems are marked with a star.

You can use any resources to solve the problems, but **you are encouraged to attempt to do the problems yourself** to best prepare for our examinations.

The standard for full credit is low: you just need to make a good-faith attempt on 4 problems.

- You can receive full credit even if your final answer is incorrect as long as you show all of your work.
- Conversely, if you just give an answer without explaining how it was derived, you may lose points.

Submission: Please handwrite your answers and show all steps in your calculations, as you would on an exam. **Submit your hard copy solutions** before the end of the day on the due date to your tutorial's homework submission box outside the 3rd floor MATH Admin Offices near Lift 25/26.

Please **coordinate with your tutorial TA directly** if you need to submit solutions electronically.

1. Define the *two-sided reduced echelon form* of a matrix A to be

$$\text{TREF}(A) := \text{RREF}(\text{RREF}(A)^\top)^\top.$$

Here $M^\top$ denotes the transpose of a matrix M .

As a warmup, calculate $\text{TREF}(A)$ if

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}.$$

Describe all the possibilities for $\text{TREF}(A)$ when A is a general $m \times n$ matrix.

2. A *column operation* on a matrix A is done by first taking the transpose of A , then doing a row operation, then taking the transpose of the result. This gives three types of column operations: swaps, rescaling, and replacement.

For each of the following cases, find a matrix M such that AM is obtained from A by

- (a) Swapping columns i and j .
 - (b) Multiplying column i by factor $c \neq 0$.
 - (c) Adding c times column i to column j when $i < j$.
 - (d) Adding c times column j to column i when $i < j$.
3. Write $A \sim B$ if B can be obtained from A by performing a sequence of row or column operations with the following constraints:
 - You can only use rescaling or replacement operations.
 - When you do a row replacement, you can only add a multiple of a row to a row farther below.
 - When you do a column replacement, you can only add a multiple of a column to a column farther to the right.

Does there exist a finite list of 3×4 matrices $M_1, M_2, M_3, \dots, M_k$ such that every 3×4 matrix A has $A \sim M_i$ for exactly one index i ? If there does, find one such list and compute k .

4. The *transpose* of a linear function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear function $g : \mathbb{R}^m \rightarrow \mathbb{R}^n$ such that

$$f(v)^\top w = v^\top g(w) \quad \text{for all } v \in \mathbb{R}^n \text{ and } w \in \mathbb{R}^m.$$

Provide answers **with justification** to the following questions:

- (a) If A is the standard matrix of f , then what is the standard matrix of the transpose of f ?

(First try this for a specific matrix A . Then find a general solution.)

- (b) Why is there only one transpose of f ?

(In other words, suppose $h : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is another transpose, and then explain why $g = h$.)

5. Find a general formula for A^n when $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and n is any positive integer.

(Hint: write $A = I + B$ where I the identity matrix. Then find a general formula for B^n first.)

6. Suppose $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbb{R}$. When is

$$A = \begin{bmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{bmatrix}$$

invertible? Find a formula for A^{-1} when A is invertible.

7. Suppose $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbb{R}$. When is

$$A = \begin{bmatrix} 0 & 0 & a_1 & b_1 \\ 0 & 0 & c_1 & d_1 \\ a_2 & b_2 & 0 & 0 \\ c_2 & d_2 & 0 & 0 \end{bmatrix}$$

invertible? Find a formula for A^{-1} when A is invertible.

8. Suppose $a, b \in \mathbb{R}$ and $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $D = \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix}$.

Find an invertible matrix P such that $AP = PD$.

Then find a general formula for A^n when n is any positive integer.

(Hint: first express A , A^2 , and A^3 in terms of P and D .)

9. (*) Suppose $a, b, c, d \in \mathbb{R}$. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ to be the function with formula

$$f(x) = ax^3 + bx^2 + cx + d.$$

Important note: we are defining f to have domain $\mathbb{R}$ and also codomain $\mathbb{R}$.

What are the possibilities for (a, b, c, d) if f is an invertible function $\mathbb{R} \rightarrow \mathbb{R}$?

(Hint: separately consider the cases when $a \neq 0$, $a = 0 \neq b$, $a = b = 0 \neq c$, etc., and use calculus.)

10. Suppose A is an $n \times n$ matrix and I is the $n \times n$ identity matrix.

Choose real numbers b and c with $b \neq 0$.

- (a) Assume $A^2 = bI + cA$. Find a formula for A^{-1} .

- (b) Suppose A is the $n \times n$ matrix with b on the main diagonal and c everywhere else.

For example, if $n = 4$ then we would have

$$A = \begin{bmatrix} b & c & c & c \\ c & b & c & c \\ c & c & b & c \\ c & c & c & b \end{bmatrix}.$$

When is A invertible? Compute A^{-1} when A is invertible.

(Your answer should be for general $b, c > 0$ and general n , not just for $n = 4$.)

11. Suppose A and B are $n \times n$ matrices such that $v^\top A w = v^\top B w \in \mathbb{R}$ for all vectors $v, w \in \mathbb{R}^n$.

Does it always hold that A and B are equal? Prove that $A = B$ or find a counterexample.

12. The *commutator* of two invertible matrices A and B is $[A, B] := ABA^{-1}B^{-1}$.

Compute a formula for $[A, B]$ when $A = \begin{bmatrix} 1 & a & b & c \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & v & w & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Here $a, b, c, d, e, v, w, x, y, z$ are arbitrary real numbers.

13. Suppose $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Find all 3×2 matrices B such that $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

14. Suppose A is an $m \times n$ matrix where $m \neq n$ and I is the $m \times m$ identity matrix.

Describe an algorithm to determine if there exists a $n \times m$ matrix B with $AB = I$, and to find such a matrix B when one does exist. Explain why your algorithm works.

15. Recall that the transpose of a matrix is formed by flipping the matrix across the northwest-to-southeast diagonal. The *anti-transpose* of a matrix is formed by flipping the matrix across the northeast-to-southwest diagonal.

Denote the anti-transpose of a matrix A as $A^\dagger$. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}^\dagger = \begin{bmatrix} 7 & 3 \\ 6 & 2 \\ 5 & 1 \end{bmatrix} \quad \text{which is not equal to} \quad \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}^\top = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \end{bmatrix}.$$

Explain why $(AB)^\dagger = B^\dagger A^\dagger$ whenever A and B are matrices such that AB is defined.