

Instructions: Choose **4 problems** and **write down detailed solutions**.

Some of the problems are more challenging than others. Harder problems are marked with a star.

You can use any resources to solve the problems, but **you are encouraged to attempt to do the problems yourself** to best prepare for our examinations.

The standard for full credit is low: you just need to make a good-faith attempt on 4 problems.

- You can receive full credit even if your final answer is incorrect as long as you show all of your work.
- Conversely, if you just give an answer without explaining how it was derived, you may lose points.

Submission: Please handwrite your answers and show all steps in your calculations, as you would on an exam. **Submit your hard copy solutions** before the end of the day on the due date to your tutorial's homework submission box outside the 3rd floor MATH Admin Offices near Lift 25/26.

Please **coordinate with your tutorial TA directly** if you need to submit solutions electronically.

1. Let $A = \begin{bmatrix} 1 & 1 & 1 & 11 \\ 6 & 8 & 16 & -26 \\ 3 & 1 & -7 & 25 \end{bmatrix}$.

- Find a basis for the column space of A .
- Find a basis for the null space of A .

Do all steps by hand and show your work.

2. Let $A = \begin{bmatrix} 2 & 3 & 16 & -5 \\ -6 & 8 & 20 & -2 \\ 3 & -2 & -8 & 3 \end{bmatrix}$.

- Find a basis for the column space of A .
- Find a basis for the null space of A .

Do all steps by hand and show your work.

3. Let $A = [a_1 \ a_2 \ \dots \ a_n]$ be a one-row matrix. Here $a_1, a_2, \dots, a_n \in \mathbb{R}$ are arbitrary numbers.

- Find a basis for the column space of A .
- Find a basis for the null space of A .

Your answer will depend on the entries in A . Remember to consider the case when $A = 0$.

4. Suppose $v_1, v_2, \dots, v_k$ are linearly independent vectors in $\mathbb{R}^n$, where $k < n$.

Describe an algorithm to find vectors $v_{k+1}, v_{k+2}, \dots, v_n$ such that $v_1, v_2, \dots, v_n$ is a basis for $\mathbb{R}^n$.

5. (*) Show that the rank one $m \times n$ matrices (the $m \times n$ matrices A with $\text{rank}(A) = \dim(\text{Col}(A)) = 1$) are precisely the matrices that can be expressed as vw^T for vectors $0 \neq v \in \mathbb{R}^m$ and $0 \neq w \in \mathbb{R}^n$.

One way to do this is by the following steps:

- Explain why $\text{rank}(vw^T) = 1$ if $0 \neq v \in \mathbb{R}^m$ and $0 \neq w \in \mathbb{R}^n$.
- Explain why a rank one matrix A must have a nonzero column.
- If rank one matrix A has a nonzero column v , explain how to find $w \in \mathbb{R}^n$ such that $A = vw^T$.

6. Let $A = \begin{bmatrix} 2 & 0 & 4 & 0 \\ 1 & 8 & 0 & 5 \\ 2 & -8 & 4 & -4 \\ 0 & 0 & x & -9 \end{bmatrix}$.

Determine the values of x such that A is invertible and find a formula for A^{-1} in this case.

7. (*) Suppose A is an $m \times n$ matrix. Let I_m and I_n be the identity matrices of size $m \times m$ and $n \times n$.

Choose **one** of the following claims and explain why it is true:

- (a) There exists a matrix B with $AB = I_m$ if and only if A has a pivot position in every row.
 (b) There exists a matrix B with $BA = I_n$ if and only if A has a pivot position in every column.

8. Suppose A is an $m \times n$ matrix and B is an $n \times q$ matrix. If $\text{rank}(A) = n$ and $\text{rank}(B) = r$, then what is $\text{rank}(AB)$ in terms of m , n , q , and r ? Justify your answer.

9. Let $v = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ and $w \in \mathbb{R}^3$. Suppose there exists a 3×3 matrix A whose null space **and** column space contains both v and w . What are the possibilities for w ? For each of these possibilities, give an example of a 3×3 matrix A whose null space and column space contains both v and w .

10. Let A be an $n \times n$ matrix. The *generalized column space* of A is the set of vectors $v \in \mathbb{R}^n$ that are in $\text{Col}(A^k)$ for **every** $k \in \{1, 2, 3, \dots\}$. Denote this subset by $\text{Col}^+ A$.

Similarly, the *generalized null space* of A is the set of vectors $v \in \mathbb{R}^n$ that are in $\text{Nul}(A^k)$ for **at least one** $k \in \{1, 2, 3, \dots\}$. Denote this subset by $\text{Nul}^+ A$.

Show that $\text{Nul}^+ A$ and $\text{Col}^+ A$ are both subspaces of $\mathbb{R}^n$.

11. (*) Continue to assume A is an $n \times n$ matrix.

Show that $v \mapsto Av$ defines an invertible function $\text{Col}^+ A \rightarrow \text{Col}^+ A$.

Using this fact or some other method, show that $\text{Nul}^+ A \cap \text{Col}^+ A = \{0\}$, or in other words, that the only vector in $\mathbb{R}^n$ in both $\text{Nul}^+ A$ and $\text{Col}^+ A$ is the zero vector.

12. (*) Show that for each $u \in \mathbb{R}^n$ there are **unique** vectors $v \in \text{Col}^+ A$ and $w \in \text{Nul}^+ A$ with $u = v + w$.
 (Use the previous exercise.)

13. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is *homogeneous of degree d* if $f(cv) = c^d f(v)$ for all $v \in \mathbb{R}^n$.

Every linear function $\mathbb{R}^n \rightarrow \mathbb{R}$ is homogeneous of degree 1.

- (a) Show that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ that is homogeneous of degree 1 must be linear.
 (b) Construct a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ that is homogeneous of degree 1 but not linear.

14. This problem is about homogeneous polynomials of degree 2.

- (a) Suppose $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$ is an $n \times n$ matrix.

Explain why the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by $f(v) = v^\top Av$ is homogeneous of degree two. Can you write this function as a polynomial in $\mathbb{R}[x_1, x_2, \dots, x_n]$?

- (b) Show that if $f \in \mathbb{R}[x_1, x_2, \dots, x_n]$ is a homogeneous polynomial of degree 2 then there is a unique *symmetric* matrix $A = A^\top$ such that $f(v) = v^\top Av$ for all $v \in \mathbb{R}^n$.

We call this matrix the *standard matrix* of f .

15. Let A be an $m \times n$ matrix.

Show that $\text{Nul}(A^\top A) = \text{Nul}(A)$ and then conclude $\text{rank}(A^\top A) = \text{rank}(A)$.