

Instructions: Choose **4 problems** and **write down detailed solutions**.

Some of the problems are more challenging than others. Harder problems are marked with a star.

You can use any resources to solve the problems, but **you are encouraged to attempt to do the problems yourself** to best prepare for our examinations.

The standard for full credit is low: you just need to make a good-faith attempt on 4 problems.

- You can receive full credit even if your final answer is incorrect as long as you show all of your work.
- Conversely, if you just give an answer without explaining how it was derived, you may lose points.

Submission: Please handwrite your answers and show all steps in your calculations, as you would on an exam. **Submit your hard copy solutions** before the end of the day on the due date to your tutorial's homework submission box outside the 3rd floor MATH Admin Offices near Lift 25/26.

Please **coordinate with your tutorial TA directly** if you need to submit solutions electronically.

1. Suppose $v_1, v_2, \dots, v_n$ and $w_1, w_2, \dots, w_n$ are vectors in $\mathbb{R}^n$. Define

$$A = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} w_1 & w_2 & \dots & w_n \end{bmatrix}.$$

Suppose A is invertible. Explain why there is a unique linear function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $f(v_i) = w_i$ for all $i = 1, 2, \dots, n$. What is the standard matrix of f ?

As an application, find the standard matrix of the unique linear function $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with

$$f\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad f\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \text{and} \quad f\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

2. Define A and B as in the previous question. Suppose A is not invertible. Explain why there is never a **unique** linear function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $f(v_i) = w_i$ for all $i = 1, 2, \dots, n$. In other words, explain why there is either no such function, or more than one.
3. A matrix A is **skew-symmetric** if $A^\top = -A$. This holds if and only if A is square with $A_{ii} = 0$ and $A_{ji} = -A_{ij}$ for all rows i and columns j that have $i \neq j$.

Suppose A is an $n \times n$ skew-symmetric matrix with all integer entries.

- (a) Prove that if n is odd then $\det(A) = 0$.
- (b) It can be shown that if n is even then $\det(A)$ is a perfect square.

For example, $A = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}$ then $\det(A) = a^2$. Check that this also holds if $n = 4$.

4. (*) Let $\mathbb{R}^{n \times k}$ be the set of $n \times k$ matrices.

The determinant is a function $\mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ that is **alternating** (switching two columns of the input multiplies by the output value by -1) and **multilinear** (it is linear as a function of the i th column of the input matrix, for any fixed column index i).

The set of alternating multilinear maps $\mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ is a vector space.

Show that this vector space is 1-dimensional with basis $\{\det\}$ in two steps:

- (a) Suppose $f : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ is an alternating multilinear function with $f(I) \neq 0$.

Explain why f must be a scalar multiple of $\det$.

- (b) Suppose $f : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ is an alternating multilinear function with $f(I) = 0$.

Explain why $f(A) = 0$ for all $A \in \mathbb{R}^{n \times n}$.

5. The set of alternating multilinear maps $\mathbb{R}^{n \times k} \rightarrow \mathbb{R}$ is a vector space for any $k \geq 1$.

Explain why this vector space is 0-dimensional if $k > n$.

6. Assume $n \geq 2$. Check that the function $\mathbb{R}^{n \times 2} \rightarrow \mathbb{R}$ given by

$$f\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \vdots \\ a_{n1} & a_{n2} \end{bmatrix}\right) = \det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right)$$

is alternating and multilinear.

Then find a basis for the vector space of alternating multilinear maps $\mathbb{R}^{n \times 2} \rightarrow \mathbb{R}$.

7. Let A be a matrix. A $k \times l$ submatrix of A is formed by choosing k (not necessarily adjacent) rows and l (not necessarily adjacent) columns and using the entries of A in these rows and columns as the corresponding entries of a $k \times l$ matrix.

The 9 different 2×2 submatrices of $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$ include $\begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix}$ and $\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$.

Explain why $\text{rank}(A) \geq k$ if A has a $k \times k$ submatrix with nonzero determinant.

8. (*) Prove that $\text{rank}(A) < k$ if every $k \times k$ submatrix of A has zero determinant.
9. Let $\mathcal{V}$ be the vector space of 3×3 matrices.

Define $L : \mathcal{V} \rightarrow \mathcal{V}$ as the linear transformation $L(A) = A - A^\top$.

- (a) Describe a basis for $\mathcal{V}$. What is $\dim(\mathcal{V})$?
- (b) Find a basis for the subspace $\mathcal{N} = \{A \in \mathcal{V} : L(A) = 0\}$. What is $\dim(\mathcal{N})$?
- (c) Find a basis for the subspace $\mathcal{R} = \{L(A) : A \in \mathcal{V}\}$. What is $\dim(\mathcal{R})$?
- (d) Find two numbers $\lambda, \mu \in \mathbb{R}$ and two nonzero matrices $A, B \in \mathcal{V}$ such that

$$L(A) = \lambda A \quad \text{and} \quad L(B) = \mu B.$$

10. (*) The set V of twice differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $f'' + f = 0$ is a vector space.

Here f' denotes the first derivative and f'' is the second derivative.

(V is a subspace of the vector space of all functions $\mathbb{R} \rightarrow \mathbb{R}$.)

Find a basis for V to compute $\dim V$.

11. (*) Suppose V is a vector space. Recall that $\text{Lin}(V, \mathbb{R})$ is the vector space of linear functions $V \rightarrow \mathbb{R}$. Choose $v \in V$. Given a linear function $f : V \rightarrow \mathbb{R}$, define

$$v^*(f) = f(v).$$

Show that if $\dim V < \infty$ then $v \mapsto v^*$ is an invertible linear function $V \rightarrow \text{Lin}(V, \mathbb{R})$.

12. (*) Alice and Bob play the following game: they start with an empty 3×3 matrix A and take turns writing numbers in each of the 9 positions. Once the matrix A is filled, Bob wins if $\det(A) = 0$ and Alice wins otherwise. Alice goes first. Which player (if any) has a winning strategy? What is this strategy?
13. A *binary matrix* is a matrix whose entries are each equal to zero or one. What is the maximum value of $\det A$ as A ranges over all 3×3 binary matrices? Find all binary matrices A that have $\det A$ equal to this maximum value. Justify why these binary matrices are the only ones whose determinants achieve the maximum value.

14. (*) Let $f(n)$ be the maximum value of $\det A$ as A ranges over all $n \times n$ matrices whose entries are each -1 or 1 . Let $g(n)$ be the maximum value of $\det A$ as A ranges over all $n \times n$ binary matrices.

What is $\frac{f(n+1)}{g(n)}$? Justify your answer.

15. Let us agree to interpret the vector $f = \begin{bmatrix} a \\ b \\ c \\ \vdots \end{bmatrix} \in \mathbb{R}^\infty$ as the polynomial $f(x) = a + bx + cx^2 + \dots$.

The subspace of vectors $f \in \mathbb{R}^\infty$ that correspond to polynomials in this way are the ones with only **finitely many nonzero entries**. Define $P \subset \mathbb{R}^\infty$ to be this subspace. The usual (but now infinite) list of elementary vectors $e_1, e_2, e_3, \dots \in \mathbb{R}^\infty$ is a basis for P .

- (a) First, write down a sensible definition of *the standard matrix* of a linear function $P \rightarrow P$.
 (b) Now find the standard matrices of the linear functions $X : P \rightarrow P$ and $D : P \rightarrow P$ defined by

$$X(f) = (\text{ the vector representing } xf(x)) \quad \text{and} \quad D(f) = (\text{ the vector representing } \frac{d}{dx}f(x)).$$

- (c) Finally, compute the standard matrix of $D \circ X - X \circ D$.