Instructions: Choose **4 problems** and **write down detailed solutions**.

Some of the problems are more challenging than others. Harder problems are marked with a star.

You can use any resources to solve the problems, but **you are encouraged to attempt to do the problems yourself** to best prepare for our examinations.

The standard for full credit is low: you just need to make a good-faith attempt on 4 problems.

- You can receive full credit even if your final answer is incorrect as long as you show all of your work.
- Conversely, if you just give an answer without explaining how it was derived, you may lose points.

Submission: Please handwrite your answers and show all steps in your calculations, as you would on an exam. **Submit your hard copy solutions** before the end of the day on the due date to your tutorial's homework submission box outside the 3rd floor MATH Admin Offices near Lift 25/26.

Please **coordinate with your tutorial TA directly** if you need to submit solutions electronically.

1. Suppose
$$V = \left\{ p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 : c_0, c_1, c_2, c_3 \in \mathbb{R} \right\}$$
is the 4-dimensional vector space of polynomials of degree ≤ 3.

   Let $T : V \to V$ be the linear map defined by
$$T(p(x)) = p(1 - x).$$

   This means that $T(1 + 2x + x^3) = 1 + 2(1 - x) + (1 - x)^3 = 4 - 5x + 3x^2 - x^3$, for example.

   Let $a_i = x^{i-1}$ and $b_i = (x + 1)^{i-1}$ for $i = 1, 2, 3, 4$.

   Then a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4 are two bases for V.

   As usual let e_1, e_2, e_3, e_4 be the standard basis of $\mathbb{R}^4$.

   There are invertible linear maps $f, g : \mathbb{R}^4 \to V$ with $f(e_i) = a_i$ and $g(e_i) = b_i$ for all $i = 1, 2, 3, 4$.

   This means that $f^{-1} \circ T \circ f$ and $g^{-1} \circ T \circ g$ and $f^{-1} \circ g$ are all linear maps $\mathbb{R}^4 \to \mathbb{R}^4$.

   Let A, B, and P be the standard matrices of $f^{-1} \circ T \circ f$, $g^{-1} \circ T \circ g$, and $f^{-1} \circ g$, respectively.

   Compute these matrices, and check that $\det(A) = \det(B)$ and $\operatorname{tr}(A) = \operatorname{tr}(B)$.

*2. Suppose V is an n-dimensional vector space and $T : V \to V$ is a linear map.

   Assume $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are two bases for V.

   As usual let $e_1, e_2, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$.

   There are invertible linear maps $f, g : \mathbb{R}^n \to V$ with $f(e_i) = a_i$ and $g(e_i) = b_i$ for all $i = 1, 2, \ldots, n$.

   This means that $f^{-1} \circ T \circ f$ and $g^{-1} \circ T \circ g$ and $f^{-1} \circ g$ are all linear maps $\mathbb{R}^n \to \mathbb{R}^n$.

   Let A, B, and P be the standard matrices of $f^{-1} \circ T \circ f$, $g^{-1} \circ T \circ g$, and $f^{-1} \circ g$, respectively.

   Find an expression for A in terms of B and P, and use this to explain why A and B have the same trace and the same determinant.

   The determinant and trace of T are **defined** to be the values of $\det(A) = \det(B)$ and $\operatorname{tr}(A) = \operatorname{tr}(B)$. To compute these values, you have to pick a basis for V, but this exercise shows that the numbers you get are the same no matter which basis you use.

3. Adopt the same setup as in the previous problem.

   Explain why it still holds that T is invertible if and only $\det(T) \neq 0$.

   In other words, explain why T is an invertible linear map if and only if the matrix A is invertible.

*4. For numbers $a, b, c, d \in \mathbb{R}$ define $\begin{bmatrix} a \\ c \end{bmatrix} \bullet \begin{bmatrix} b \\ d \end{bmatrix} = ab + cd$. For $v \in \mathbb{R}^2$ define $\|v\| = \sqrt{v \bullet v}$.

   Suppose $v_1, v_2 \in \mathbb{R}^2$ are nonzero vectors. This exercise walks through a proof of the sometimes useful identity $v_1 \bullet v_2 = \|v_1\|\|v_2\| \cos(\theta)$ where θ is the angle between v_1 and v_2.

   To define the angle θ precisely, assume that rotating v_1 counterclockwise by θ radians gives a **positive** scalar multiple of v_2, and that v_1 and v_2 are labeled such that this angle has $\theta \in [0, \pi]$. (This means that if we have to go more than π radians counterclockwise from v_1 to get to v_2, then we switch the names of the vectors.)

   (a) Define u_1 and u_2 to be the unit vectors in the directions of v_1 and v_2, so $u_i = \frac{1}{\|v_i\|} v_i$.

   Explain why $v_1 \bullet v_2 = \|v_1\|\|v_2\|(u_1 \bullet u_2)$.

   (b) Suppose $\psi \in [0, 2\pi)$ and $M = \begin{bmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{bmatrix}$. Check that $(Mu_1) \bullet (Mu_2) = u_1 \bullet u_2$.

   (c) Explain why you can choose a value of ψ such that $Mu_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $Mu_2 = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$ where θ is the angle between v_1 and v_2. Conclude that $u_1 \bullet u_2 = \cos(\theta)$.

   Combining all three parts tells us that $v_1 \bullet v_2 = \|v_1\|\|v_2\| \cos(\theta)$.

5. Suppose A is a square matrix with all entries in $\mathbb{R}$.

   Let $a, b \in \mathbb{R}$ and suppose $\lambda = a + bi \in \mathbb{C}$ is an eigenvalue for A.

   Show that if $A^\top = -A$ then $a = 0$.

6. Suppose A is a square matrix with all entries in $\mathbb{R}$.

   Let $a, b \in \mathbb{R}$ and suppose $\lambda = a + bi \in \mathbb{C}$ is an eigenvalue for A.

   Show that if $A^\top = A$ then $b = 0$.

7. Suppose A is a square matrix with all entries in $\mathbb{R}$.

   Let $a, b \in \mathbb{R}$ and suppose $\lambda = a + bi \in \mathbb{C}$ is an eigenvalue for A.

   Show that if $A^\top = A^{-1}$ then $a^2 + b^2 = 1$.

*8. Suppose A is a square matrix with all entries in $\mathbb{C}$. Determine whether the eigenvalue properties in the Exercises 5, 6, and 7 still hold. That is, for each of the three statements, either find a counterexample or show that the given property for λ is still true when A has complex entries.

9. Define

$$\mathbf{1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad i = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad j = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad k = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

   As with complex numbers, when $a, b, c, d \in \mathbb{R}$ we abbreviate by setting

$$a + bi + cj + dk = a\mathbf{1} + bi + cj + dk = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}.$$

   Now consider the real vector space of *quaternionic numbers* $\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$.

(a) Compute all nine products yz for $y, z \in \{i, j, k\}$.

Conclude that $\mathbb{H}$ is closed under multiplication, that is, if $y, z \in \mathbb{H}$ then $yz \in \mathbb{H}$.

However, multiplication in $\mathbb{H}$ is not commutative (as it is in $\mathbb{R}$ and $\mathbb{C}$).

(b) Why is multiplication in $\mathbb{H}$ associative? (In the sense that $x(yz) = (xy)z$ for all $x, y, z \in \mathbb{H}$.)

*10. Suppose $z = a + bi + cj + dk \in \mathbb{H}$. Find a formula for $\det(z)$.

Then find a formula for z^{-1} when $z \neq 0$ and check that this element is still in $\mathbb{H}$.

*11. Suppose $p(z) = a_n z^n + \cdots + a_2 z^2 + a_1 z + a_0$ is polynomial with $n > 0$, $a_0, \ldots, a_n \in \mathbb{C}$, and $a_n \neq 0$.

Two different versions of the fundamental theorem of algebra:

(A) There are complex numbers $r_1, r_2, \ldots, r_n \in \mathbb{C}$ such that $p(z) = a_n(z - r_1)(z - r_2)\cdots(z - r_n)$.

(B) There is a complex number $r \in \mathbb{C}$ with $p(r) = 0$.

Prove that these two statements are equivalent.

**12. Suppose $p(z) = a_n z^n + \cdots + a_2 z^2 + a_1 z + a_0$ is polynomial with $n > 0$, $a_0, \ldots, a_n \in \mathbb{C}$, and $a_n \neq 0$.

This exercise walks through a proof of the claim that

- There is a complex number $r \in \mathbb{C}$ with $p(r) = 0$.

In the previous exercise we saw that this is equivalent to the *Fundamental Theorem of Algebra*.

This exercise is fairly long, but if you want to understand why this fundamental theorem is true, it may be interesting!

(a) Warmup: what number works for r if we have $a_0 = 0$?

For the rest of this exercise assume $0 \neq a_0 \in \mathbb{C}$.

For $0 < N \in \mathbb{R}$ define $C(N) \subseteq \mathbb{R}^2$ to be the set of points (x, y) where x is the real part of

$$p(N \cdot \cos\theta + N \cdot \sin\theta \cdot i)$$

and y is the imaginary part, as $\theta \in \mathbb{R}$ varies from 0 to 2π.

$C(N)$ is a closed curve in xy-plane that starts and ends at the point $(x, y) = (\mathsf{Re}(p(N)), \mathsf{Im}(p(N)))$.

Define $W(N)$ to be the number of times $C(N)$ travels counter-clockwise completely around the origin $(x, y) = (0, 0)$ as θ varies from 0 to 2π. This called the *winding number* of $p(z)$.

(b) Explain why for any positive integer n we have

$$(N \cdot \cos\theta + N \cdot \sin\theta \cdot i)^n = N^n \cdot \cos(n\theta) + N^n \cdot \sin(n\theta) \cdot i.$$

Use the fact that $N \cdot \cos\theta + N \cdot \sin\theta \cdot i$ is N times a 2×2 rotation matrix.

(c) Suppose $p(z) = z^n + 1$.

Deduce that $C(N)$ draws a circle of radius N^n centered at $(x, y) = (1, 0)$, which travels n times around its center point as θ varies from 0 to 1.

In this case, what is $W(N)$ if $0 < N < 1$ and what is $W(N)$ if $N > 1$?

Draw a picture of $C(N)$ for each of these two cases.

(d) Suppose $p(z) = z^4 - 4z^3 + 5$.

Find a way to draw $C(N)$ and compute $W(N)$ where $N = 0.1$, 1, 2, 10, and 100.

Do some experiments to convince yourself that when $N \approx 0$ is very small the curve $C(N)$ is always very close to the nonzero point $(x, y) = (5, 0)$ so $W(N) = 0$, while if $N \gg 0$ is very large then $C(N)$ looks like a circle of radius N^4 that travels 4 times around the origin so $W(N) = 4$.

Moreover, as N increases, the value of $W(N)$ can only change if there is real number $N > 0$ such that the curve $C(N)$ passes through the origin.

(e) Suppose for a general $p(z)$ the curve $C(N)$ contains the origin $(0, 0)$.

Explain why this means there is some $r \in \mathbb{C}$ with $p(r) = 0$.

The observations you made in (b) for the given polynomial $p(z)$ are completely generic.

For any $p(z)$, when $N \approx 0$ is very small the curve $C(N)$ is very close to the nonzero point

$$(x, y) = (\mathsf{Re}(a_0), \mathsf{Im}(a_0))$$

so $W(N) = 0$, while if $N \gg 0$ is very large then $C(N)$ looks like a circle of radius N^n centered at the same point, which travels n times around the origin so $W(N) = n$.

But in order for $W(N)$ to change value, there must be some $N > 0$ such that $(0, 0) \in C(N)$. Then, using (e), we deduce that there is some $r \in \mathbb{C}$ with $p(r) = 0$.