Instructions: Choose **4 problems** and **write down detailed solutions**.

Some of the problems are more challenging than others. Harder problems are marked with a star.

You can use any resources to solve the problems, but **you are encouraged to attempt to do the problems yourself** to best prepare for our examinations.

The standard for full credit is low: you just need to make a good-faith attempt on 4 problems.

- You can receive full credit even if your final answer is incorrect as long as you show all of your work.
- Conversely, if you just give an answer without explaining how it was derived, you may lose points.

Submission: Please handwrite your answers and show all steps in your calculations, as you would on an exam. **Submit your hard copy solutions** before the end of the day on the due date to your tutorial's homework submission box outside the 3rd floor MATH Admin Offices near Lift 25/26.

Please **coordinate with your tutorial TA directly** if you need to submit solutions electronically.

1. Do there exist two linearly independent vectors in $\mathbb{R}^4$ that are orthogonal to all three of the vectors

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ -1 \\ 2 \\ 5 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 7 \\ 1 \\ 4 \\ 7 \end{bmatrix}?$$

   Find two such vectors if they exist, and otherwise explain why there are no such vectors.

2. Consider the plane $P = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : 3x - 2y + 6z = 0 \right\}$ in $\mathbb{R}^3$.

   (a) The subspace P is 2-dimensional. Find an orthogonal basis for P.

   (b) Find the vector in P that is closest to $v = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

3. Find an orthonormal basis for the subspace of vectors of the form

$$\begin{bmatrix} a + 3b + 2c \\ 3a + 2b + 4c \\ 2a + 5b + 4c \\ 6a + 5b + 4c \end{bmatrix}$$

   where $a, b, c \in \mathbb{R}$ are real numbers.

4. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \\ -1 & 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 6 \\ 5 \\ 5 \end{bmatrix}$.

   Find the orthogonal projection of b onto the orthogonal complement of the column space of A.

   Show all the steps in your derivation to receive credit.

5. Suppose $A = A^\top$ is a symmetric $n \times n$ matrix with all real entries.

   (a) Explain why $v^\top A v > 0$ for all nonzero $v \in \mathbb{R}^n$ if A has all positive eigenvalues.

   (b) Explain why $v^\top A v = 0$ for some nonzero $v \in \mathbb{R}^n$ if A has zero as an eigenvalue.

   (c) Explain why $v^\top A v < 0$ for some $v \in \mathbb{R}^n$ if A has a negative eigenvalue.

This shows that $v^\top Av > 0$ for all nonzero $v \in \mathbb{R}^n$ if and only if A has all positive eigenvalues.

A symmetric matrix with this property is called *positive definite*.

This also shows that $v^\top Av \geq 0$ for all $v \in \mathbb{R}^n$ if and only if A has all nonnegative eigenvalues.

A symmetric matrix with this property is called *positive semidefinite*.

6. Find the values of $x, y \in \mathbb{R}$ that minimize the distance between the vectors $\begin{bmatrix} x \\ x \\ x \end{bmatrix}$ and $\begin{bmatrix} y \\ 3y \\ -1 \end{bmatrix}$.

   Show all the steps in your derivation to receive credit.

7. Suppose A is an $n \times n$ matrix. **Assume A is diagonalizable.** Explain why $A^\top$ is similar to A.

8. Let A be a square $n \times n$ matrix whose rows are orthonormal.

   Explain why the columns of A are orthonormal.