Instructions: Choose **4 problems** and **write down detailed solutions**.

Some of the problems are more challenging than others. Harder problems are marked with a star.

You can use any resources to solve the problems, but **you are encouraged to attempt to do the problems yourself** to best prepare for our examinations.

The standard for full credit is low: you just need to make a good-faith attempt on 4 problems.

- You can receive full credit even if your final answer is incorrect as long as you show all of your work.
- Conversely, if you just give an answer without explaining how it was derived, you may lose points.

Submission: Please handwrite your answers and show all steps in your calculations, as you would on an exam. **Submit your hard copy solutions** before the end of the day on the due date to your tutorial's homework submission box outside the 3rd floor MATH Admin Offices near Lift 25/26.

Please **coordinate with your tutorial TA directly** if you need to submit solutions electronically.

1. Find all least-squares solutions to the linear equation $Ax = b$ where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}.$$

   Show all the steps in your derivation to receive credit.

2. Suppose $A = \begin{bmatrix} 1 & 3 \\ 0 & -1 \\ 2 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 1 \\ h \end{bmatrix}$.

   (a) For which values of $h \in \mathbb{R}$ does $Ax = b$ have an exact solution?

   For such h, find all exact solutions to $Ax = b$.

   (b) For which values of $h \in \mathbb{R}$ does $Ax = b$ have a least-squares solution?

   For such h, find all least-squares solutions to $Ax = b$.

   Show all the steps in your derivation to receive credit.

3. Suppose $A = \begin{bmatrix} 3 & -6 & 0 \\ -6 & 0 & 6 \\ 0 & 6 & -3 \end{bmatrix}$.

   Find an orthogonal matrix U and a diagonal matrix D such that $A = UDU^\top$.

   Show all the steps in your derivation to receive credit.

4. Let A be a matrix with all real entries.

   (a) Suppose A is diagonal.

   What can you say about $\mathsf{RREF}(A)$? Describe $\mathsf{RREF}(A)$ as explicitly as possible.

   (b) Suppose A is invertible.

   What can you say about $\mathsf{RREF}(A)$? Describe $\mathsf{RREF}(A)$ as explicitly as possible.

   (c) Suppose the columns of A are orthonormal.

   What can you say about $\mathsf{RREF}(A)$? Describe $\mathsf{RREF}(A)$ as explicitly as possible.

(d) Suppose the columns of A are orthogonal.

What can you say about $\mathsf{RREF}(A)$? Describe $\mathsf{RREF}(A)$ as explicitly as possible.

(e) Suppose A is $n \times n$ with n distinct eigenvalues.

What can you say about $\mathsf{RREF}(A)$? Describe $\mathsf{RREF}(A)$ as explicitly as possible.

(f) Suppose A is symmetric.

What can you say about $\mathsf{RREF}(A)$? Describe $\mathsf{RREF}(A)$ as explicitly as possible.

5. Find an orthogonal matrix U and a diagonal matrix D such that

$$A = UDU^T \quad \text{for the matrix } A = \begin{bmatrix} 2 & 4 & 4 \\ 4 & 8 & 8 \\ 4 & 8 & 8 \end{bmatrix}.$$

6. Suppose $A = A^\top$ is a symmetric $n \times n$ matrix with all real entries.

Explain why $A = B^\top B$ for some $n \times n$ matrix B (possibly with complex entries).

Compute a value for B when $A = \begin{bmatrix} 3 & -6 & 0 \\ -6 & 0 & 6 \\ 0 & 6 & -3 \end{bmatrix}$.

7. Suppose $A = A^\top$ is a positive semidefinite symmetric $n \times n$ matrix with all real entries.

Recall that this means $v^\top A v \geq 0$ for all $v \in \mathbb{R}^n$.

Explain how to find a positive semidefinite symmetric $n \times n$ matrix B with all real entries such that

$$A = B^2.$$

Compute B when $A = \begin{bmatrix} 40 & -28 & -26 \\ -28 & 52 & 2 \\ -26 & 2 & 25 \end{bmatrix}$.

8. (*) Suppose $a_1 < a_2 < a_3$ are real numbers and $b_1, b_2, b_3 \in \mathbb{R}$.

Explain why the points $(x, y) = (a_1, b_1), (a_2, b_2), (a_3, b_3)$ are all on the same line if and only if

$$a_1(b_3 - b_2) + a_2(b_1 - b_3) + a_3(b_2 - b_1) = 0.$$

(One approach: consider the line of best fit through the three points.)

9. (*) Suppose A is a 2×2 matrix.

(a) Can you always find an invertible matrix B such that $BAB^{-1} = A^\top$?

(b) Can you always find an invertible symmetric matrix $C = C^\top$ such that $CAC^{-1} = A^\top$?

In each case, **explain how to construct B and C** (if they exist).