Instructions: Choose **4 problems** and **write down detailed solutions**.

Some of the problems are more challenging than others. Harder problems are marked with a star.

You can use any resources to solve the problems, but **you are encouraged to attempt to do the problems yourself** to best prepare for our examinations.

The standard for full credit is low: you just need to make a good-faith attempt on 4 problems.

- You can receive full credit even if your final answer is incorrect as long as you show all of your work.
- Conversely, if you just give an answer without explaining how it was derived, you may lose points.

Submission: Please handwrite your answers and show all steps in your calculations, as you would on an exam. **Submit your hard copy solutions** before the end of the day on the due date to your tutorial's homework submission box outside the 3rd floor MATH Admin Offices near Lift 25/26.

Please **coordinate with your tutorial TA directly** if you need to submit solutions electronically.

1. Find a singular value decomposition for the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 3 & 0 \end{bmatrix}$$

   Show all steps in your derivation (but feel free to check your answer with a calculator).

2. Find a singular value decomposition for the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

   Show all steps in your derivation (but feel free to check your answer with a calculator).

3. Find a singular value decomposition for the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 4 & 0 \\ 0 & 1 \end{bmatrix}.$$

   Show all steps in your derivation (but feel free to check your answer with a calculator).

4. Find a singular value decomposition for the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

   Show all steps in your derivation (but feel free to check your answer with a calculator).

5. Suppose A is a 2×2 matrix with a singular value decomposition

$$A = U\Sigma V^T$$

   where U and V are orthogonal 2×2 matrices and

$$\Sigma = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}.$$

The first column of U is the vector $\begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$.

Draw a picture of the region in $\mathbb{R}^2$ given by

$$\left\{ Ax : x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \text{ is a vector with } x_1^2 + x_2^2 \leq 1 \right\}.$$

6. Suppose $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$ are orthonormal vectors. Assume $k < n$.

   (a) Describe an algorithm to find $n - k$ vectors

   $$v_{k+1}, v_{k+2}, \ldots, v_n \in \mathbb{R}^n$$

   such that $v_1, v_2, \ldots, v_n$ is an orthonormal basis for $\mathbb{R}^n$.

   (b) Suppose $n = 3$ and $v_1 = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$.

   Find two vectors $v_2, v_3 \in \mathbb{R}^3$ such that v_1, v_2, v_3 is an orthonormal basis for $\mathbb{R}^n$.

*8. Suppose A is an $m \times n$ matrix with at most one nonzero entry in each row and column.

Describe a singular value decomposition for A.

Use the following notation in your answer: suppose the positions of A with nonzero entries are

$$(i_1, j_1), (i_2, j_2), \ldots, (i_r, j_r)$$

where $j_1 < j_2 < \cdots < j_r$, and the entries in these positions are $a_1, a_2, \ldots, a_r \in \mathbb{R}$.

(It may be useful to compare your answer with #2 and #5.)

*9. Suppose A is an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$.

Suppose σ_k is much larger than σ_{k+1} where $k < \operatorname{rank}(A)$.

Describe an algorithm using an SVD for A that produces a good rank k approximation to A.

Apply this algorithm (using a computer or calculator) to find a rank 2 approximation to the matrix

$$A = \begin{bmatrix} 0.2 & 0.1 & -0.1 \\ 1.2 & 0.1 & 0.8 \\ 1.0 & -2.0 & 5.5 \end{bmatrix}.$$

*10. Suppose A is an invertible 3×3 matrix with a singular value decomposition

$$A = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^\top.$$

Give a geometric interpretation of the numbers $\sigma_1, \sigma_2, \sigma_3$ and the vectors $u_1, u_2, u_3, v_1, v_2, v_3 \in \mathbb{R}^3$.

Your answer should involve the sphere $\mathbb{S} = \{w \in \mathbb{R}^3 : \|w\| \leq 1\}$ and its image $A\mathbb{S} = \{Aw : w \in \mathbb{S}\}$.

11. Give your favorite proof that $\operatorname{rank}(A) = \operatorname{rank}(A^\top)$ for any $m \times n$ matrix A.

*12. Let A be an $m \times n$ matrix. Describe an algorithm to find a unit vector $v \in \mathbb{R}^n$ such that

$$\|Av\| = \max\{\|Ax\| : x \in \mathbb{R}^n \text{ with } \|x\| = 1\}.$$

*13. Prove that any square matrix A is similar to its transpose $A^\top$.