MATH 2121 – Linear Algebra

Lecture # 1

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↳ contact: use email
Discord
canvas

All course information on public website tutorials start in week #2

1 Introduction

Course website: https://www.math.ust.hk/~emarberg/Math2121/

Each lecture corresponds to one or more sections in the textbook.

Today's lecture corresponds to Section 1.1.

For more discussion of the topics in any lecture, see the textbook.

first due date end of week #2

Syllabus: HW 10% { 5% online webwork, 5% offline practice problems }

* drop lowest scores

* no due date extensions

2hr

~Oct 21 (week 7) ← Midterm 30%

Final 60%

3hr

exams: closed notes, no calculators

Today: setup notation, define linear system

matrices and matrix notation → solving linear systems

Throughout, we'll be using the following notation:

- $\mathbb{R}$ denotes the real numbers: any number representable on a computer in decimal notation
- $\mathbb{Q}$ denotes the rational numbers p/q. $(p, q \in \mathbb{Z})$
- $\mathbb{Z}$ denotes the integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
- $\mathbb{N}$ denotes the nonnegative integers $\{0, 1, 2, \ldots\}$.

Ellipsis ("...") notation: we write $a_1, a_2, \ldots, a_7$ instead of the full list $a_1, a_2, a_3, a_4, a_5, a_6, a_7$.

We use the same convention to write $a_1, a_2, \ldots, a_n$ even when n is a variable integer.

2 Systems of linear equations

Choose a $\boxed{\text{positive integer } n.}$ → usually $n = 2, 3,$ or 4

Let $x_1, x_2, \dots, x_n$ be variables. Let $a_1, a_2, \dots, a_n, b$ be numbers in $\mathbb{R}$.

Unlike in calculus, where our favorite variables are x, y, z, here we prefer $x_1, x_2, x_3, \dots$.

$n = 3$: $4x_1 + 2x_2 - x_3 = 7$

$a_1 = 4$, $a_2 = 2$, $a_3 = -1$

$b = 7$

Definition. We refer to

$$\boxed{a_1x_1 + a_2x_2 + \cdots + a_nx_n = b}$$

as a *linear equation* in the variables $x_1, x_2, \dots, x_n$.

Notation. Another way of writing this equation is $\boxed{\sum_{i=1}^{n} a_ix_i = b.}$

$2x_2 + 4x_1 - 7 = x_3$

$x_3 + 7 = 4x_1 + 2x_2$

(consider to be same equation)

The symbol "$\sum$" is the Greek letter sigma, for "sum."

There are many other ways of writing the same equation. For example:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n - b = 0,$$

$$b = a_1x_1 + a_2x_2 + \cdots + a_nx_n,$$

$$a_1x_1 + a_2x_2 + a_3x_3 = b - a_4x_4 - a_5x_5 - \cdots - a_nx_n,$$

and so forth. We consider all of these equations to be the same thing.

is case when

$a_1 = a_2 = a_3 = 0$

$b = 0$

Example. The following are all linear equations in the variables x_1, x_2, x_3:

$$\boxed{3x_1 = 2x_2,} \qquad \boxed{3x_1 + \tfrac{4}{3}x_2 - \sqrt{2}x_3 = 7,} \qquad \boxed{0 = 0,} \qquad \boxed{0 = 1.}$$

case when

$a_1 = a_2 = a_3 = 0$

$b = 1$

Even though the last two equations involve no variables, they have the form required. *The last equation is false, but a false equation is still an equation.*

The following are *not* linear equations in the variables x_1, x_2, x_3:

$$3x_1^2 + 4x_2 = 7, \qquad x_1x_2 = x_3, \qquad 2^{x_1} = x_2, \qquad \underbrace{\sqrt{x_1^2} = x_2}.$$

exponent $\neq 1$

product of variable

variable in an exponent

$\sqrt{x_1^2} = |x_1| \neq x_1$

in some set of variables

A *system of linear equations* or *linear system* is a list of linear equations like:

$$\boxed{\begin{aligned} 2x_1 - x_2 + \sqrt{3}x_3 &= 8 \\ x_1 - 4x_3 &= 8 \\ x_2 &= 0 \end{aligned}}$$

linear system of 3 equations in 3 variables

For any linear system, there is always a set of associated variables, usually $x_1, x_2, \ldots, x_n$ for some n. The example above has 3 equations in the variables x_1, x_2, x_3.

Not every equation needs to involve every variable x_i. In fact, it could happen the some variable x_i appears in none of the equations.

Definition. A *solution* of a linear system in variables $x_1, x_2, \ldots, x_n$ is a list of n numbers $(s_1, s_2, \ldots, s_n)$ with the property that if we set $x_1 = s_1$, $x_2 = s_2$, $\ldots$, $x_n = s_n$ in our equations, we get all true statements.

A solution $(s_1, s_2, \ldots, s_n)$ is *nonzero* if at least one number $s_i \neq 0$.

Two linear systems are *equivalent* if they have the same set of solutions.

usually not so easy to determine when this holds

in the same set of variables

Example. How many solutions can a linear system have?

add eqs to cancel variables

1. The system

   2 eqs
   2 vars

$$\left\{\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}\right.$$

$$\Rightarrow (x_1 - 2x_2) + (-x_1 + 3x_2) = -1 + 3$$

$$\Rightarrow \boxed{x_2 = -1 + 3 = 2}$$

$$\Rightarrow x_1 - 2x_2 = x_1 - 2(2) = x_1 - 4 = -1$$

$$\Rightarrow \boxed{x_1 = 3}$$

   exactly

   has $\boxed{\text{one solution}}$ $(s_1, s_2) = (3, 2)$.

2. But the system

   2 eqs
   2 vars

$$\left\{\begin{aligned} x_1 - 2x_2 &= -1 \\ 3x_1 - 6x_2 &= -3 \end{aligned}\right. \Leftrightarrow \left\{ x_1 - 2x_2 = -1 \right.$$

   "equivalent to"

   infinitely

   has $\boxed{\text{many solutions:}}$ $(s_1, s_2) = (1, 1)$ or $(3, 2)$ or $(5, 3)$ or $\ldots$.

3. Whereas the system

   2 eqs
   2 vars

$$\left\{\begin{aligned} x_1 - 2x_2 &= -1 \\ x_1 - 2x_2 &= 0 \end{aligned}\right.$$

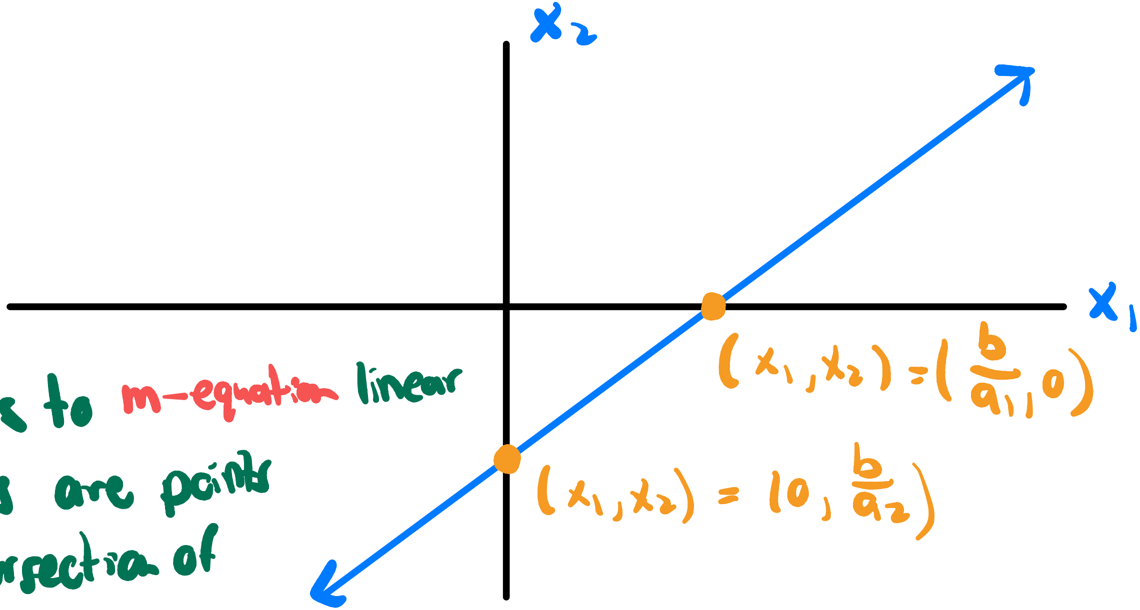
   different right sides

   same left side

   has $\boxed{\text{no solutions.}}$

the set of solutions to a single linear equation

$a_1 x_1 + a_2 x_2 = b$ is a line in the plane:



solutions to m-equation linear systems are points of intersection of m lines

every point on this line is a solution $(x_1, x_2) = (s_1, s_2)$

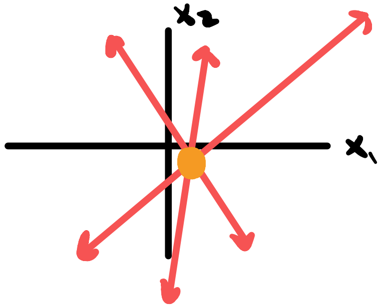
Theorem. A linear system in two variables x_1 and x_2 has either 0, 1, or infinitely many solutions.

Proof by geometry. A solution to one equation $ax_1 + bx_2 = c$ represents a point on a line, viewing the pair (x_1, x_2) as a point in the Cartesian plane.

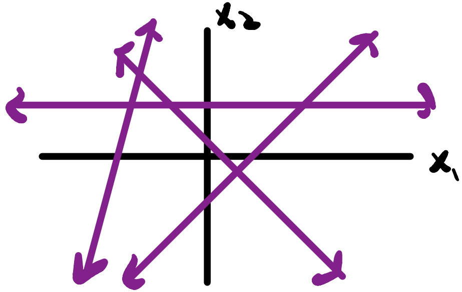
A solution to a system of 2-variable linear equations represents a point where the lines defined by the equations all intersect.

But a collection of lines all intersect at either 0 points (they don't have a common intersection), 1 point (the unique point of intersection) or at infinitely many points (in the case when the lines are all *the same line*). $\square$

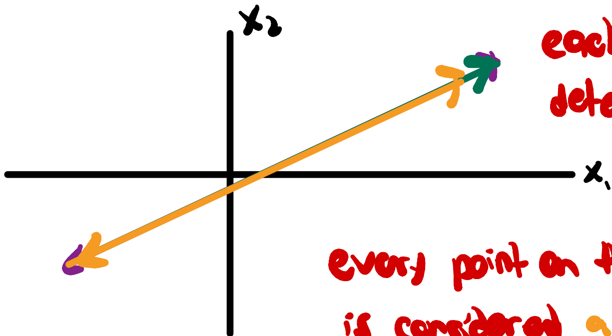
1 solution



0 solutions



infinitely many (∞ - many) solutions



each eq determines the same line

every point on this line is considered a point of intersection

Theorem. A linear system in two variables x_1 and x_2 has either 0, 1, or infinitely many solutions.

Proof by algebra. We just need to check that if our linear system has two different solutions, then it has infinitely many solutions.

Given a linear system, define the associated *homogeneous system* to be the linear system in which each equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ is replaced by

$$\boxed{a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0.}$$

For example the homogeneous system of

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3 \end{cases} \quad \text{is} \quad \begin{cases} x_1 - 2x_2 = 0 \\ -x_1 + 3x_2 = 0. \end{cases}$$

* If (s_1, s_2) is a solution to our starting system and (t_1, t_2) is a solution to the associated homogeneous system, then $(s_1 + t_1, s_2 + t_2)$ is also a solution to our starting system.

$\hookrightarrow$ if $\begin{cases} a_1 s_1 + a_2 s_2 = b \\ a_1 t_1 + a_2 t_2 = 0 \end{cases}$ then $a_1(s_1+t_1) + a_2(s_2+t_2) = b+0 = b$

This is because if $a_1s_1 + a_2s_2 = b$ and $a_1t_1 + a_2t_2 = 0$ then

$$a_1(s_1 + t_1) + a_2(s_2 + t_2) = (a_1s_1 + a_2s_2) + (a_1t_1 + a_2t_2) = b + 0 = b.$$

**

But if the homogeneous system has a nonzero solution (t_1, t_2), then it has infinitely many solutions: the pairs $(2t_1, 2t_2)$, $(3t_1, 3t_2)$, $(4t_1, 4t_2)$, $\ldots$ are all solutions, as

$$\text{if } a_1t_1 + a_2t_2 = 0 \text{ then } a_1(4t_1) + a_2(4t_2) = 4(a_1t_1 + a_2t_2) = 4 \cdot 0 = 0. \checkmark$$

Therefore: if our starting system has a solution and the homogeneous system has a nonzero solution, then starting system has infinitely many solutions.

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Finally, if our starting system has two different solutions (s_1, s_2) and (r_1, r_2), then

diff : $$(t_1, t_2) = (s_1 - r_1, s_2 - r_2)$$ solves homogeneous system

is a nonzero solution to homogeneous system, as if $a_1s_1 + a_2s_2 = b$ and $a_1r_1 + a_2r_2 = b$ then $a_1(s_1 - r_1) + a_2(s_2 - r_2) = \underbrace{(a_1s_1 + a_2s_2)}_{=b} - \underbrace{(a_1r_1 + a_2r_2)}_{=b} = b - b = 0.$ $\square$

* Algebraic argument generalizes to linear systems with any number of variables n

* Theorem is special to linear systems

For example $(x-2)(x-3) = \overbrace{x^2-5x+6=0}^{\text{non-linear eq.}}$

has exactly two solutions $x=2$ or $x=3$

but does not have ∞-many solutions

A linear system is *consistent* if it has at least one solution, and *inconsistent* if it has zero solutions.

Both the algebraic and geometric proofs generalize to any number of variables:

Theorem. A linear system (in any number of variables) has either 0, 1, or infinitely many solutions.

plural of "matrix"

3 Matrices

A *matrix* is just a rectangular array of numbers, like these ones:

$$\begin{bmatrix} 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1/7 \\ 0.2 \\ 3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 5 & 3 \\ 2 & \pi \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 7 & 6 & -4 & 3 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 0 & 11 \end{bmatrix}.$$

row

column

zero matrix is $\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$

has rows and columns

We denote a general matrix by

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \end{bmatrix}$$

3 x 4 matrix

Here "A_{23}" is pronounced "A, two, three". in position (3,2)

This matrix is 3-by-4: it has 3 rows and 4 columns.

One says that a matrix A is m-by-n or $m \times n$ if it has m rows and n columns.

We usually write $\boxed{A_{ij}}$ (pronounced "A, i, j") for the entry in the ith row and jth column of a matrix A.

Position (i,j) of a matrix

means entry in row i and column j

Matrices are useful as a compact way of writing a linear system.

Consider the linear system

$$\left\{\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}\right.$$

The $\boxed{\textit{augmented matrix}}$ of this system is

RHS of the eqs

$$\begin{array}{c}\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \\ \begin{array}{cccc} x_1 & x_2 & x_3 & \text{RHS of eq} \end{array}\end{array} \quad \text{which we sometimes write as} \quad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right].$$

We consider both of these to be the same matrix. The $|$ on the right is just there to remind us that this is an augmented matrix rather than a coefficient matrix.

Example

$$\left.\begin{aligned} 2x_1 + x_4 &= -1 \\ x_2 + 3x_3 &= 5 \end{aligned}\right\} \longleftrightarrow \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \\ \left[\begin{array}{cccc|c} 2 & 0 & 0 & 1 & -1 \\ 0 & 1 & 3 & 0 & 5 \end{array}\right] \end{array}$$

augmented matrix

The $\boxed{\textit{coefficient matrix}}$ of a linear system is given by dropping the last column of the augmented matrix.

For the linear system

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ 5x_1 - 5x_3 = 10 \end{cases}$$

the *coefficient matrix* is $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$.

We tend to work with the augmented matrix more than the coefficient matrix.

Exercise: how would you generalize these definitions to any linear system?

coeff matrix of a linear system with $\begin{matrix} m \text{ eqs} \\ n \text{ vars} \end{matrix}$ has size $m \times n$

augmented matrix has size $m \times (n+1)$

4 Solving linear systems

We solve linear systems by adding equations together to cancel variables.

Next class, we will discuss a general algorithm for doing these cancellations that turns the augmented matrix of a linear system into a description of its solutions.

The following example contains some of the main ideas in this algorithm.

Example. To solve

augmented matrix

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right]$$

$-5x_1 + 10x_2 - 5x_3 = 0$

$5x_1 - 5x_3 = 10$

we first add -5 times equation 1 to equation 3 to get

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ \boxed{10x_2 - 10x_3 = 10} & \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{array}\right].$$

We then multiply equation 2 by $1/2$ to get

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ 10x_2 - 10x_3 &= 10 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 10 & -10 & 10 \end{array}\right].$$

We then add -10 times equation 2 to equation 3:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ 30x_3 &= -30 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 30 & -30 \end{array}\right].$$

$-10x_2 + 40x_3 = 40$

Multiple equation 3 by 1/30:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= -1 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right].$$

The augmented matrix of the last system is *triangular*: all entries in positions (i, j) with $i > j$ are zero.

Remember that i is the row, j is the column.

We can solve for x_1, x_2, x_3 from a triangular system, working from the bottom up:

- The last equation $\boxed{x_3 = -1}$ is already as simple as possible.
- Substitute into second equation: $x_2 - 4x_3 = x_2 - 4(-1) = 4 \Rightarrow \boxed{x_2 = 0}$.
- Substitute into first equation: $x_1 - 2x_2 + x_3 = x_1 - 2(0) + (-1) = 0 \Rightarrow \boxed{x_1 = 1}$.

so unique solution is $(x_1, x_2, x_3) = (1, 0, -1)$

Definition. In solving this system of equations, we performed the following *(elementary)* $\boxed{\textit{row operations}}$ on the augmented matrix of the system:

1. Replacement: replace a row by the sum of itself and a multiple of another row:
$$\begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 \\ 6 & 8 \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 501 & 602 \\ 5 & 6 \end{bmatrix}.$$

2. Scaling: multiply a row by a *nonzero* number:
$$\begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 \\ -500 & -600 \end{bmatrix}.$$

3. Interchange: swap two rows: $\begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix}.$

Two matrices are $\boxed{\textit{row equivalent}}$ if one can be transformed to the other by a sequence of zero or more row operations.

Each row operation is reversible. why?

	inverse op
- scale a row by c to	- scale by $1/c$
- swap rows i and j	- do same swap
- replace row i by c times row j plus row i	- replace row i by $-c$ times row j plus row i

$$\begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix} \xrightarrow{c=100} \begin{bmatrix} 501 & 602 \\ 5 & 6 \end{bmatrix} \xrightarrow{c=-100} \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix}$$

Theorem. If the augmented matrices of two linear systems are row equivalent, then the systems are equivalent (i.e., have same solutions).

Proof. If we have two linear systems that differ by a single row operation, then a given list of numbers $(s_1, s_2, \ldots, s_n)$ is a solution to one system if and only if it is a solution to the other system.

Thus performing a sequence of row operations does not change the set of solutions to a linear system. $\square$