# MATH 2121 — Lecture #3

---

Plan:

1. Review, RREF(A), solving linear systems
2. Vector notation + definitions
3. Linear combinations + span

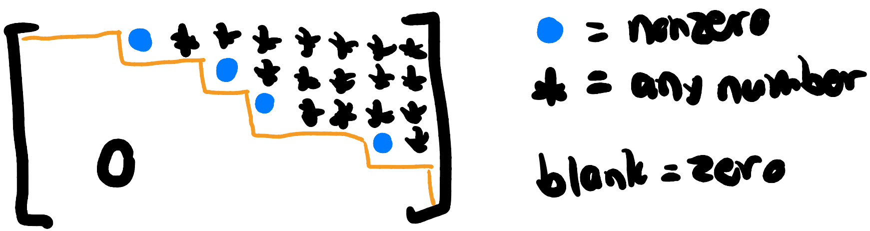
# 1 Last time: reduction to (reduced) echelon form

The *leading entry* in a nonzero row of a matrix is the first nonzero entry going from left to right. For example, the row $\begin{bmatrix} 0 & 0 & \boxed{7} & 0 & 5 \end{bmatrix}$ has leading entry 7, which occurs in the 3rd column.

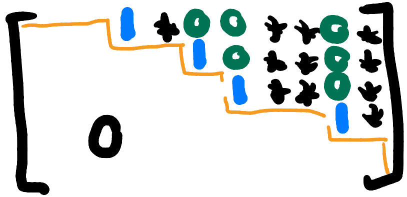
**Definition.** A matrix is in *echelon form* if it has the following properties:

1. If a row is nonzero, then every row above it is also nonzero.
2. Leading entries are arranged strictly left to right as rows go top to bottom.
3. Every entry below a leading entry in the same column is zero.

Picture :



Picture :



● = nonzero
* = any number
blank = zero

In echelon form: $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

nonzero rows

zero rows

out of order ✗

Not in echelon form: $\begin{bmatrix} 1 & 2 & 0 & 0 \\ \boxed{0 \quad 0 \quad 0 \quad 0} \\ 0 & 3 & 5 & 0 \end{bmatrix}$ ✗, $\begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ✗, $\begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 3 & 5 & 0 \\ 0 & 0 & 4 & 5 \end{bmatrix}$.

**Definition.** A matrix is in *reduced echelon form* if

1. The matrix is in echelon form.
2. Each nonzero row has leading entry 1.
3. The leading 1 in each nonzero row is the only nonzero number in its column.

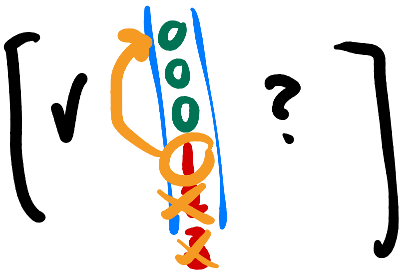
**Theorem.** A matrix $A$ is row equivalent to a single matrix in reduced echelon form.

We denote this matrix by $\boxed{\mathsf{RREF}(A)}$.

meaning, can be transformed using row operations: ① replacement ② row swaps ③ rescaling

We have an algorithm to compute RREF(A)

- proceed left to right, starting with first nonzero column
- goal at each step: move a nonzero entry to top of current column and then use row ops to cancel entries below



- finally convert from echelon form to reduced form.

The *row reduction algorithm* is a way of constructing $\mathsf{RREF}(A)$ from $A$.

This algorithm is something you should memorize.

(not all steps show)

**Example.** Writing $\to$ to indicate a sequence of row operations, we have

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 1 \\ 1 & 3 & 9 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 2 & 8 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \mathrm{RREF}(A)$$

an echelon form of A

and the last matrix is the reduced echelon form of the first matrix.

# Going from RREF(A) to solutions of a linear system

Consider the nonzero rows of RREF$(A)$. In these row find the first nonzero entry from left to right.

If one of these leading entries is in column $j$, then $j$ is a *pivot column* of $A$.

For example if

$$\text{RREF}(A) = \begin{bmatrix} \boxed{1} & -2 & 0 & 3 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

then leading entries are in positions $(1, 1)$ and $(2, 3)$ so pivot columns are 1 and 3.

If $A$ is the augmented matrix of a linear system in variables $x_1, x_2, \ldots, x_n$, then we say that $x_i$ is a *basic variable* if $i$ is a pivot column of $A$ and that $x_i$ is a *free variable* if $i$ is not a pivot column of $A$.

not basic

linear system $\rightarrow$ augmented matrix A $\rightarrow$ RREF(A) $\rightarrow$ another (simpler) system w/ same solutions

To determine the basic and free variables of the system, we have to perform the row reduction algorithm to figure out what $\mathsf{RREF}(A)$ is first.

Once we have done this, we can conclude that:

- The system has 0 solutions if the last column is a pivot column of $A$.
- The system has infinitely many solutions if the last column is not a pivot column but there is at least one free variable.
- The system has exactly 1 solution otherwise (if there are no free variables, and the last column is not a pivot column).

linear system $\rightarrow$ A $\rightarrow$ RREF(A) $\rightarrow$ pivot columns

RREF(A) $\rightarrow$ basic variables

RREF(A) $\rightarrow$ free variable

Assuming last column of A is NOT a pivot column

Moreover, here's how you can solve the system: write down the equations in the linear system whose augmented matrix is $\mathsf{RREF}(A)$. Each nontrivial equation starts with a basic variable $x_i$ and has the form

basic $x_i$ + ( an expression involving free variables ) = ( a number )

After moving the expression involving free variables to the right side we get

basic $x_i$ = ( a number ) − ( an expression involving free variables ).

To form the general solution to our original linear system, we just choose arbitrary values for the free variables and express the basic variables using these equations.

in terms of free variable

**Example.** The linear system

$$\begin{cases} 3x_2 - 6x_3 = 6 \\ 3x_1 - 7x_2 + 8x_3 = -5 \\ 3x_1 - 9x_2 + 12x_3 = -9 \end{cases}$$

has augmented matrix

$$A = \left[\begin{array}{ccc|c} 0 & 3 & -6 & 6 \\ 3 & -7 & 8 & -5 \\ 3 & -9 & 12 & -9 \end{array}\right]$$

Its reduced echelon form is

$$\mathsf{RREF}(A) = \left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

basic : $x_1, x_2$

free : $x_3$

This means that the pivot columns of $A$ are columns 1 and 2, so $x_1$ and $x_2$ are basic variables while $x_3$ is a free variable. The last column is not a pivot column, so the linear system has $\boxed{\text{infinitely many solutions}}$.

↳ have one free variable

The linear system with augmented matrix $\mathsf{RREF}(A)$ is

$$\begin{cases} x_1 - 2x_3 = 3 \\ x_2 - 2x_3 = 2 \\ \quad\quad\quad 0 = 0 \end{cases} \quad \text{which we can rewrite as} \quad \begin{cases} x_1 = 3 + 2x_3 \\ x_2 = 2 + 2x_3 \\ 0 = 0. \end{cases}$$

basic

free

We choose an arbitrary value for the free variable $x_3 = a \in \mathbb{R}$.

Then the general solution is $(x_1, x_2, x_3) = (3 + 2a, 2 + 2a, a)$ where $a$ is any number.

general solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3+2a \\ 2+2a \\ a \end{bmatrix}$$

where $a \in \mathbb{R}$ is any number

last col not pivot
as all other columns are pivots

- no free vars
- all variables basic

**Corollary.** Suppose a linear system has augmented matrix $A$.

If $\mathsf{RREF}(A)$ has the form $\left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & b_1 \\ 0 & 1 & \cdots & 0 & b_2 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & b_m \end{array}\right]$ where all blank entries are zero, then the linear system has exactly one solution, and this solution is given by

$$\boxed{(x_1, x_2, \ldots, x_m) = (b_1, b_2, \ldots, b_m).}$$

*Proof.* The starting linear system has the same solutions as the linear system whose augmented matrix is $\mathsf{RREF}(A)$. But the second system consists of the equations $x_1 = b_1,\ x_2 = b_2,\ \ldots,\ x_m = b_m$. $\square$

$$\begin{cases} x_1 = b_1 \\ x_2 = b_2 \\ \vdots \\ x_m = b_m \end{cases}$$

better notation to work with linear systems

start to let us think about geometry of the solutions to a linear system

## 2 Vectors

Until we discuss vector spaces, the term *vector* will always refer to a matrix with exactly one column:

matrix with one column

$$\begin{bmatrix} 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \sqrt{7} \\ \sqrt{6} \end{bmatrix}.$$

Sometimes people refer to vectors defined like this as *column vectors*.

We write a general vector as $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ where each $v_i$ is a real number.

the entry of $v$ in row $i$

Two vectors $u$ and $v$ are equal if they have the same number of rows and the same entries in each row.

two vector operations:

addition:

$$\begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \\ 10 \\ 11 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} 1-1 \\ 3-2 \\ 0+10 \\ 2+11 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 10 \\ 13 \end{bmatrix}$$

rescaling
("scalar multiplication")

$$5\begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} 5\cdot 1 \\ 5\cdot 3 \\ 5\cdot 0 \\ 5\cdot 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \\ 0 \\ 10 \end{bmatrix}$$

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \text{ and } u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

The *size* of a vector is its number of rows. We can add two vectors of the same size:

$$v + u = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}.$$

If $u$ and $v$ don't have the same size then $u + v$ is not defined.

If $v$ is a vector and $c \in \mathbb{R}$ is a *scalar*, i.e., a real number, then we define

$$cv = \quad cv = c \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}.$$

We call the new vector $cv$ the *scalar multiple* of $v$ by $c$.

↳ "scalar" just means number

combing vector addition + scalar mult. $\rightsquigarrow \begin{cases} \text{vector subtraction} \\ \text{vector negation} \end{cases}$

For example, we have

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \quad \text{and} \quad \boxed{-\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.}$$

We define *subtraction* of vectors as addition after multiplying by the scalar $-1$:

$$\boxed{\begin{bmatrix} 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ -7 \end{bmatrix}.} = \begin{bmatrix} 1-1 \\ -2-5 \end{bmatrix}$$

negation:

$$-v = \begin{bmatrix} -v_1 \\ -v_2 \\ \vdots \\ -v_n \end{bmatrix} = (-1)v$$

subtraction:

$$v - u = v + (-1)u$$

Caution: $u+v$ and $u-v$ are ONLY defined if $u$ and $v$ have same size

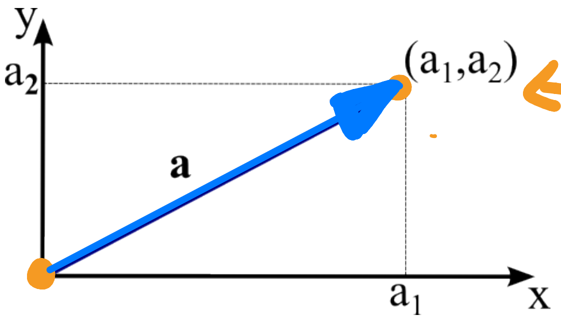
Notation: for each integer $n>0$ define $\mathbb{R}^n$ to be the set of all $n$-row vectors

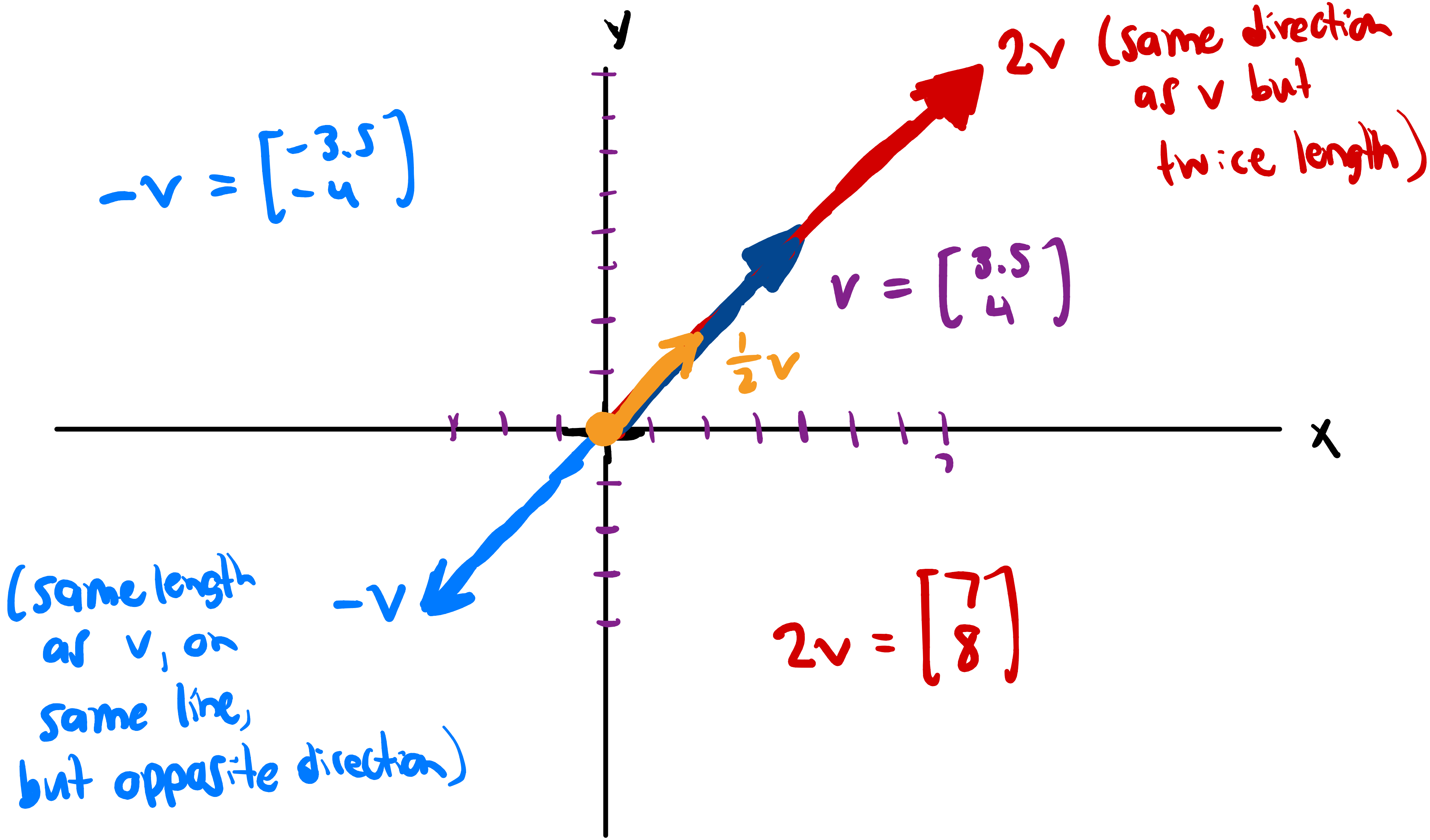
formally: $\mathbb{R}^n = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \text{ for any numbers } v_1, v_2, \ldots, v_n \in \mathbb{R} \right\}$

good geometric intuition for $\begin{cases} \mathbb{R}^1 = \mathbb{R} = \text{number line} \\ \mathbb{R}^2 = xy\text{-plane} \\ \mathbb{R}^3 = 3d\text{-space} \end{cases}$

Draw vector $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$ as arrow in x-y-plane starting at origin

We write $\mathbb{R}^n$ for the set of all vectors with exactly $n$ rows. Vectors $\boxed{a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2}$ are drawn as arrows in the $xy$-plane from the origin to the point $(x, y) = (a_1, a_2)$:



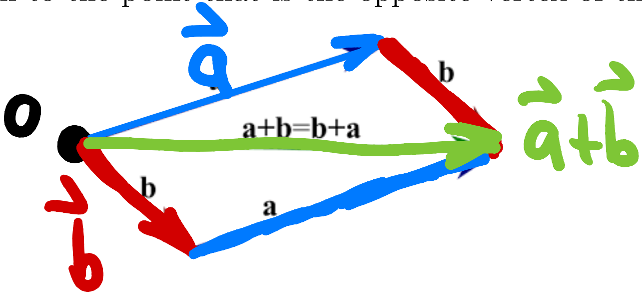



Scalar multiplication by $c \in \mathbb{R}$ $\longleftrightarrow$ rescaling arrow length $(c > 0)$
or reversing direction $(c < 0)$

given vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^2$

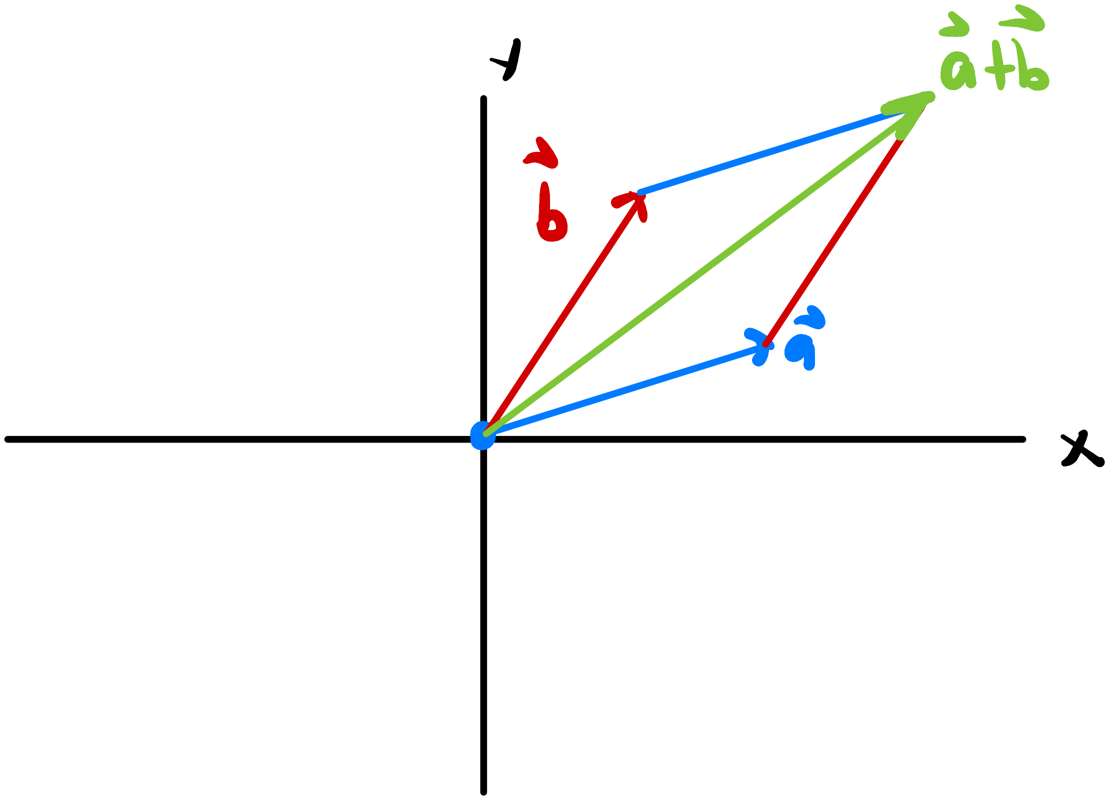
then $\vec{a}+\vec{b}$ is the diagonal of the parallelogram with sides $\vec{a}$ and $\vec{b}$

**Proposition.** The sum $a+b$ of two vectors $a, b \in \mathbb{R}^2$ is the vector represented by the arrow from the origin to the point that is the opposite vertex of the parallelogram with sides $a$ and $b$:




if $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ then

$$\vec{a}+\vec{b} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \end{bmatrix}$$



y
x
$\vec{b}$
$\vec{a}$
$\vec{a}+\vec{b}$

another viewpoint:

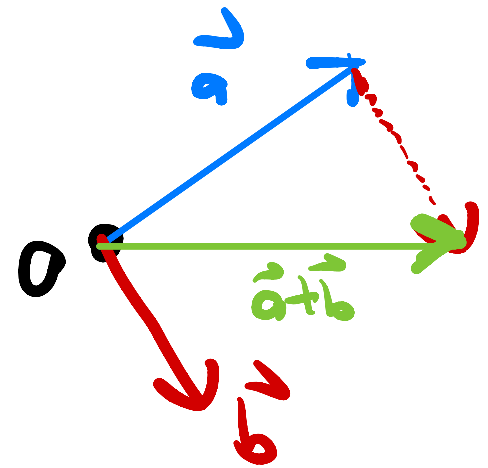
$\vec{a}+\vec{b}$ is formed by starting at origin,

following $\vec{a}$ to its endpoint,

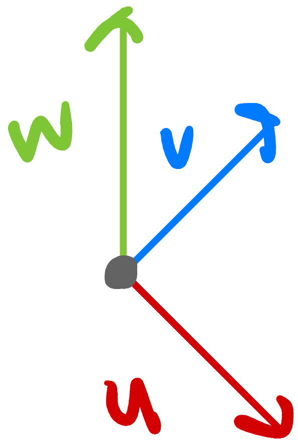
then attaching a copy of $\vec{b}$

and following it to its endpoint

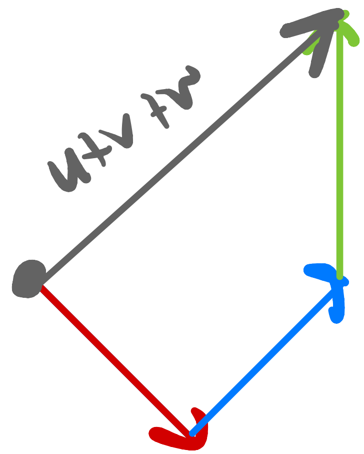
then drawing an arrow from origin to here



this also works with 3 or more vectors:




u+v+w =



The *zero vector* in $\mathbb{R}^n$ is the vector $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ whose entries are all zero.

We use the same symbol "0" to mean both the number zero and the zero vector in $\mathbb{R}^n$ for any $n$. You may have to use context to figure out which number or zero vector "0" means in a given expression.

We have $\boxed{0 + v = v + 0 = v}$ for any vector $v$.

vectors

**Definition.** Suppose $\boxed{v_1, v_2, \ldots, v_p \in \mathbb{R}^n}$ are vectors and $c_1, c_2, \ldots, c_p \in \mathbb{R}$.

We often refer to the numbers $c_i$ as *scalars*.

The vector $\boxed{y = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p}$ is called a $\boxed{\textit{linear combination}}$ of $v_1, v_2, \ldots, v_p$.

We say that $y$ is "the linear combination of $v_1, v_2, \ldots, v_p$ with *coefficients* $c_1, c_2, \ldots, c_p$."

a new vector formed from some given vectors by doing vector additions and scalar multiplications

$\begin{bmatrix}4\\22\\33\end{bmatrix}$ is a linear combination of $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}, \begin{bmatrix}7\\0\\0\end{bmatrix}$

because
$$\underset{y}{\begin{bmatrix}4\\22\\33\end{bmatrix}} = 5\underset{v_1}{\begin{bmatrix}1\\2\\3\end{bmatrix}} + 3\underset{v_2}{\begin{bmatrix}2\\4\\6\end{bmatrix}} - \underset{v_3}{\begin{bmatrix}7\\0\\0\end{bmatrix}}$$

$c_1 = 5 \quad c_2 = 3 \quad c_3 = -1$

$\begin{bmatrix}0\\0\\0\end{bmatrix}$ is also a lin. comb of these vectors $0 = 0v_1 + 0v_2 + 0v_3$

$x_1 a + x_2 b = x_1\begin{bmatrix}1\\-2\\-5\end{bmatrix} + x_2\begin{bmatrix}2\\5\\6\end{bmatrix} = \begin{bmatrix}x_1+2x_2\\-2x_1+5x_2\\-5x_1+6x_2\end{bmatrix} = \begin{bmatrix}7\\4\\-3\end{bmatrix} = c$

new question:

**Example.** Suppose $a = \begin{bmatrix}1\\-2\\-5\end{bmatrix}$ and $b = \begin{bmatrix}2\\5\\6\end{bmatrix}$ and $c = \begin{bmatrix}7\\4\\-3\end{bmatrix}$.

Is $c$ a linear combination of $a$ and $b$?

If it were, we could find numbers $\boxed{x_1, x_2 \in \mathbb{R} \text{ such that } x_1 a + x_2 b = c,}$ so that

$$\begin{cases} x_1 + 2x_2 = 7 \\ -2x_1 + 5x_2 = 4 \\ -5x_1 + 6x_2 = -3. \end{cases}$$

To determine if this linear system has a solution, use row reduction:

$$A = \begin{bmatrix}1 & 2 & 7\\-2 & 5 & 4\\-5 & 6 & -3\end{bmatrix} \to \begin{bmatrix}1 & 2 & 7\\0 & 9 & 18\\0 & 16 & 32\end{bmatrix} \to \begin{bmatrix}1 & 2 & 7\\0 & 1 & 2\\0 & 1 & 2\end{bmatrix} \to \begin{bmatrix}1 & 2 & 7\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix} \to \mathsf{RREF}(A) = \begin{bmatrix}1 & 0 & 3\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix}.$$

The pivot columns of $A$ are 1 and 2: the last column is **not** a pivot column, so our linear system is consistent, which means that $c$ **is** a linear combination of $a$ and $b$.

Yes!

in fact RREF(A) $\Leftrightarrow \begin{cases} x_1 = 3 \\ x_2 = 2 \\ 0 = 0 \end{cases} \Leftrightarrow \boxed{c = 3a + 2b}$

$x_i$ variable

$a_i$ vector in $\mathbb{R}^m$

$b \in \mathbb{R}^m$

We generalize this example with the following statement.

**Proposition.** A vector equation of the form $\boxed{x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = b}$ where $x_1, x_2, \ldots, x_n$ are variables and $a_1, a_2, \ldots, a_n, b \in \mathbb{R}^m$ are vectors, has the same solutions as the linear system with augmented matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 & \ldots & a_n & b \end{bmatrix}. \qquad (**)$$

$m \times (n+1)$ matrix

This notation means the matrix whose $i$th column is $a_i$ and last column is $b$.

In other words, the vector $b$ is a linear combination of $a_1, a_2, \ldots, a_n$ if and only if the linear system whose augmented matrix is $(**)$ is consistent.

if and only if last column of RREF(A) has no pivot

span is all linear combinations you can form from given vectors

**Definition.** The *span* of a list of vectors $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$ is the set of all vectors $y \in \mathbb{R}^n$ that are linear combinations of $v_1, v_2, \ldots, v_p$. We denote this set by

$$\boxed{\mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_p\}} \qquad \text{or} \qquad \text{span}\{v_1, v_2, \ldots, v_p\}.$$

Another more direct way to define the span is by the formula

$$\mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_p\} = \{\underline{a_1 v_1 + a_2 v_2 + \cdots + a_p v_p} : a_1, a_2, \ldots, a_p \in \mathbb{R}\}.$$

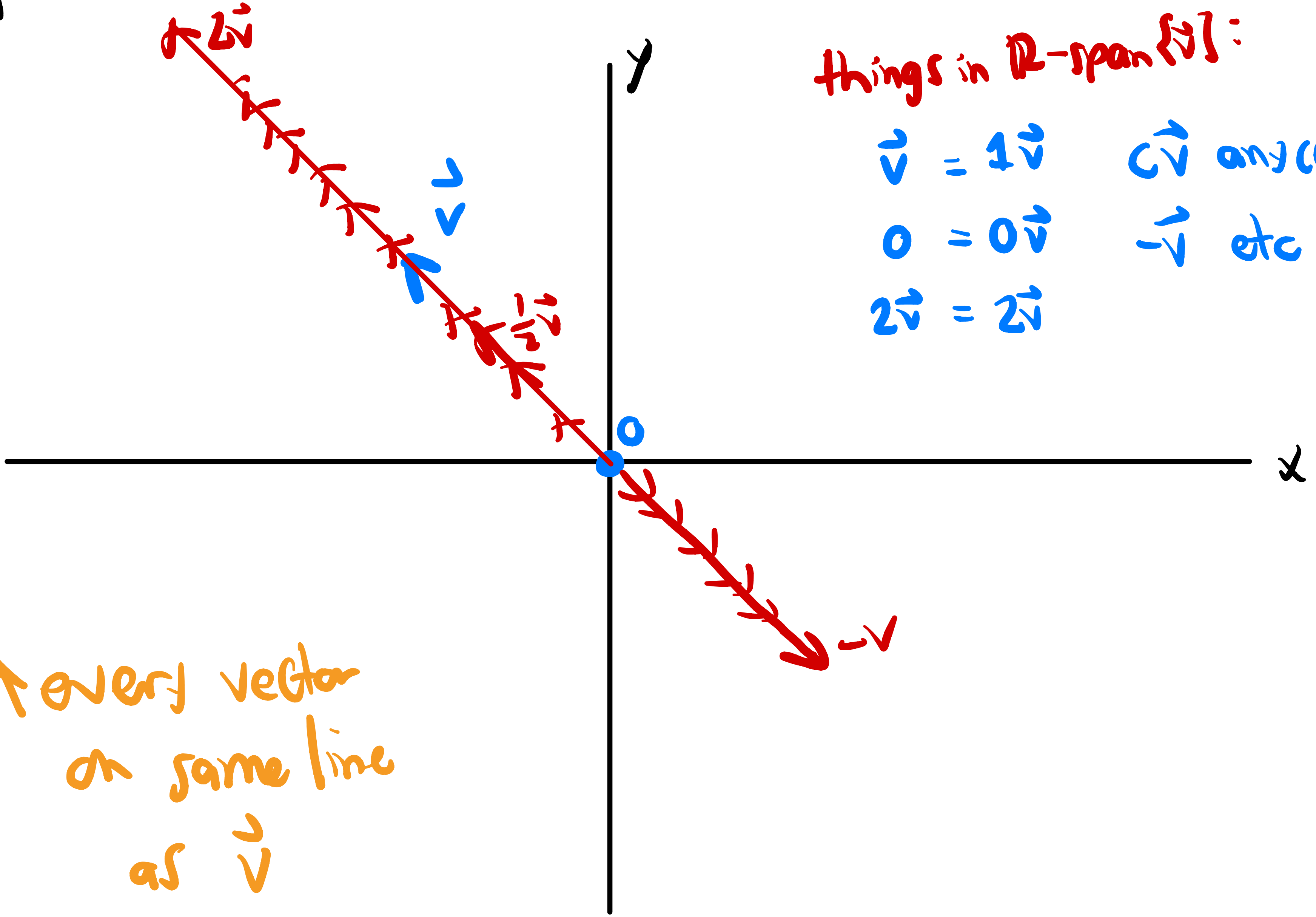
**Example.** The span of $0 \in \mathbb{R}^n$ is just $\boxed{\mathbb{R}\text{-span}\{0\} = \{a0 : a \in \mathbb{R}\} = \{0\}.}$

(This is the only case where the span is not an infinite set.)

The span of $0 \neq v \in \mathbb{R}^n$ is set $\boxed{\mathbb{R}\text{-span}\{v\} = \{av : a \in \mathbb{R}\}}$ of all scalar multiples of $v$.

(This set is infinite. It contains $v$, $-v$, $0 = 0v$, $2v$, $-2v$, $\frac{36475}{7}v$, $-\sqrt{107}v$, $\pi v$, etc.)

Picture of $\mathbb{R}$-span$\{\vec{v}\}$ when $\begin{bmatrix}0\\0\end{bmatrix} \neq \vec{v} \in \mathbb{R}^2$




things in $\mathbb{R}$-span$\{\vec{v}\}$:

$\vec{v} = 1\vec{v}$ $\quad$ $c\vec{v}$ any $c \in \mathbb{R}$

$0 = 0\vec{v}$ $\quad$ $-\vec{v}$ etc

$2\vec{v} = 2\vec{v}$

every vector on same line as $\vec{v}$

things in span(v,w) include: v+w = 1v+1w, w = 0v+1w
v = 1v+0w   0 = 0v+0w
−v = −v+0w
2v+3w etc.

The span of two nonzero vectors $v, w \in \mathbb{R}^n$ is $\boxed{\mathbb{R}\text{-span}\{v, w\} = \{av + bw : a, b \in \mathbb{R}\}}$.

- If $v$ is a scalar multiple of $w$ then $\mathbb{R}\text{-span}\{v, w\} = \mathbb{R}\text{-span}\{w\}$.
- If $w$ is a scalar multiple of $v$ then $\mathbb{R}\text{-span}\{v, w\} = \mathbb{R}\text{-span}\{v\}$.

Outside these cases, we have no simpler way of describing $\mathbb{R}\text{-span}\{v, w\}$ except as "the set of all vectors of the form $av + bw$ where $a$ and $b$ are any real numbers."

**Corollary.** If $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$, then a vector $y \in \mathbb{R}^n$ belongs to

$$\mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_p\}$$

same as saying

$y$ is a linear comb. of $\vec{v}_1, \ldots, \vec{v}_p$

if and only if the $n \times (p + 1)$ matrix

$$\begin{bmatrix} v_1 & v_2 & \ldots & v_p & y \end{bmatrix}$$

is the augmented matrix of a consistent linear system.

$$\mathbb{R}\text{-span}\{cw, w\} = \{\underbrace{acw + bw}_{(ac+b)w} \mid a, b \in \mathbb{R}\} = \mathbb{R}\text{-span}\{w\}$$

For most mathematical operations, we have separate words for the operation itself, the verb that does the operation, and the result of doing the operation.

For example the $+$ operation is called "addition", the corresponding verb is "add", and the result of adding is called a "sum".

$\mathbb{R}$-span is also a mathematical operation, though its inputs and outputs are **sets of vectors**. But confusingly, the words that go along with $\mathbb{R}$-span are all the same.

For example, choose some vectors $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$ and let $S = \mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_p\}$.

- We refer to the operation $\mathbb{R}$-span as the *span*.
- The output $S$ of $\mathbb{R}$-span applied to $v_1, v_2, \ldots, v_p$ is the *span* of $v_1, v_2, \ldots, v_p$.
- Sometimes we use *span* as a verb and say that $v_1, v_2, \ldots, v_p$ *span* the set $S$.
- Finally, saying a vector $w \in \mathbb{R}^n$ *is spanned by* $v_1, v_2, \ldots, v_p$ means that $w \in S$.

What does $\mathbb{R}$-span$\{v_1, v_2, \ldots, v_p\}$ look like? in $\mathbb{R}^2$, can describe as one of 3 shapes

- We can visualize the span of the 0 vector as a single point.
- We imagine the span of vectors on the same line through the origin as that line.
- In $\mathbb{R}^2$, if the span of $v_1, v_2, \ldots, v_p$ is not a line, then it is the whole plane.