MATH 2121 — Lecture #4

1. Review: vectors, lin. comb., span
2. Matrix-vector multiplication Av
3. Matrix equations

1 Last time: Vectors

matrix w/ 1 column

A *(column)* $\boxed{\textit{vector}}$ of size n is an $n \times 1$ matrix: $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$.

Let $\boxed{\mathbb{R}^n}$ be the set of all vectors with exactly n rows.

+ We can add two vectors of the same size: $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$.

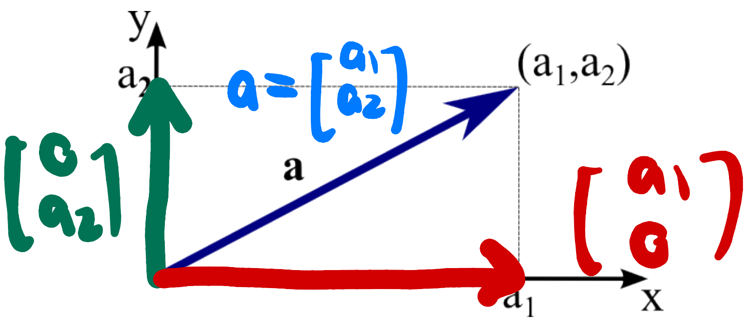
two vector operations: + and $\cdot$

$$\begin{bmatrix}1\\2\\0\end{bmatrix}+\begin{bmatrix}3\\5\\7\end{bmatrix}=\begin{bmatrix}4\\7\\7\end{bmatrix}\in\mathbb{R}^3 \qquad 6\cdot\begin{bmatrix}1\\3\\0\\2\end{bmatrix}=6\begin{bmatrix}1\\3\\0\\2\end{bmatrix}=\begin{bmatrix}6\\18\\0\\12\end{bmatrix}\in\mathbb{R}^4$$

"number"

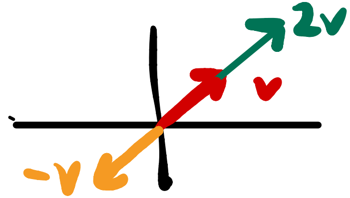
- We can multiply a vector by a $\boxed{\textit{scalar}}$: $cv = c\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}=\begin{bmatrix}cv_1\\cv_2\\\vdots\\cv_n\end{bmatrix}$ for $c\in\mathbb{R}$.

We draw $a=\begin{bmatrix}a_1\\a_2\end{bmatrix}\in\mathbb{R}^2$ as arrow in Cartesian plane from origin to $(x,y)=(a_1,a_2)$:

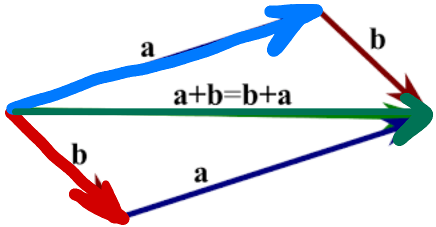


$$\begin{bmatrix}a_1\\a_2\end{bmatrix}=\begin{bmatrix}a_1\\0\end{bmatrix}+\begin{bmatrix}0\\a_2\end{bmatrix}$$

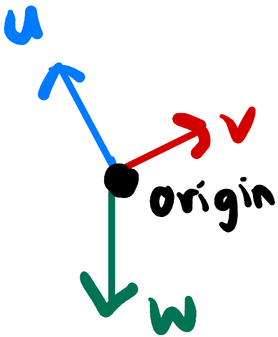
scalar multiplication :



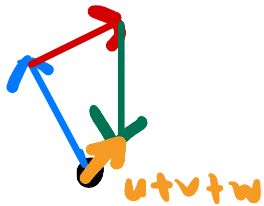
Relative to this picture, the sum $a + b$ of two vectors $a, b \in \mathbb{R}^2$ is the arrow from the origin to the opposite vertex of the parallelogram with sides a and b:



The *zero vector* $0 \in \mathbb{R}^n$ is the vector $0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. We always have $0 + v = v + 0 = v$.



what is u + v + w ?



linear system $\longleftrightarrow$ augmented matrix $\longleftrightarrow$ vector equation

$$\begin{cases} 2x_1 + 3x_2 = 7 \\ 4x_1 + 5x_2 = 0 \\ x_1 + 2x_2 = 1 \end{cases} \qquad \left[\begin{array}{cc|c} 2 & 3 & 7 \\ 4 & 5 & 0 \\ 1 & 2 & 1 \end{array}\right] \qquad x_1 \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix}$$

$$[\, v_1 \;\; v_2 \,|\, b \,] \qquad\qquad v_1 \qquad v_2 \qquad b$$

have same solutions

several vectors of SAME size

A $\boxed{\textit{linear combination}}$ of vectors $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$ is any vector of the form

$\boxed{y = c_1v_1 + c_2v_2 + \cdots + c_pv_p \in \mathbb{R}^n}$ for any choice of numbers $c_1, c_2, \ldots, c_p \in \mathbb{R}$.

The $\boxed{\textit{span}}$ of vectors $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$ is the set of all of their linear combinations.

(usually infinite, hard to describe)

We denote this set by

$$\boxed{\mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_p\}} \qquad \text{or} \qquad \text{span}\{v_1, v_2, \ldots, v_p\}.$$

Proposition. If $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$, then a vector $y \in \mathbb{R}^n$ belongs to $\mathbb{R}$-span$\{v_1, v_2, \ldots, v_p\}$ if and only if the $n \times (p+1)$ matrix $\left[\ \underbrace{v_1 \quad v_2 \quad \ldots \quad v_p}_{A} \quad y\ \right]$ is the augmented matrix of a consistent linear system.

is y is span{x₁, x₂, ..., xp}?

does RREF(A) have no pivot in last column?

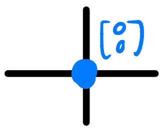
same answer

Ex $\underset{v_1}{2\begin{bmatrix}1\\2\\4\end{bmatrix}}+3\underset{v_2}{\begin{bmatrix}3\\5\\0\end{bmatrix}}+(-2)\underset{v_3}{\begin{bmatrix}1\\1\\1\end{bmatrix}}=\underset{y}{\begin{bmatrix}9\\17\\6\end{bmatrix}}$

y is a linear comb. of v_1 v_2 v_3

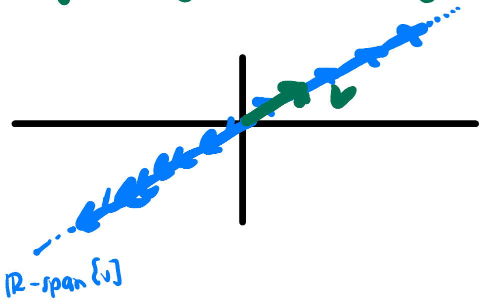
In terms of geometry, the span of a set of vectors in $\mathbb{R}^2$ is either a point (at the origin), a line (through the origin), or the whole plane $\mathbb{R}^2$.

point: $\mathbb{R}\text{-span}\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\} = \left\{c\begin{bmatrix}0\\0\end{bmatrix} \;\middle|\; c\in\mathbb{R}\right\} = \left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$



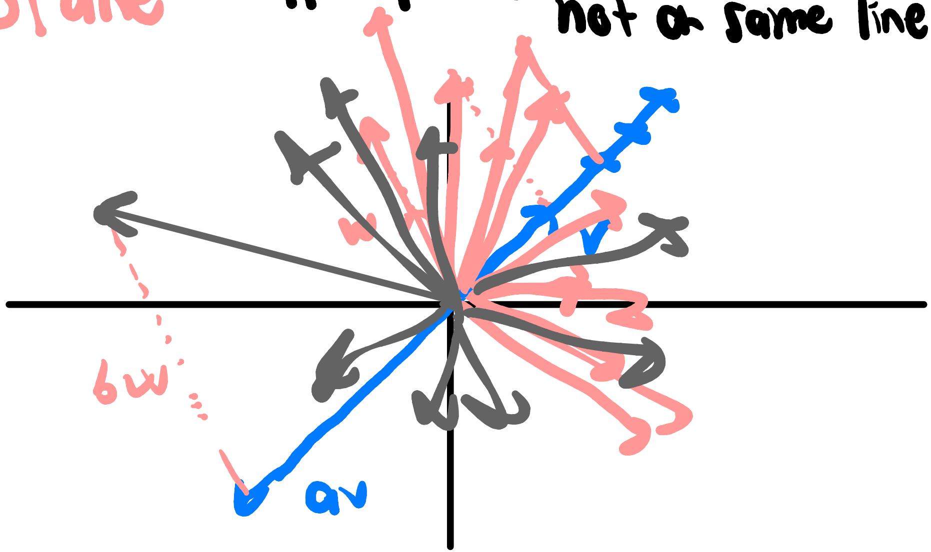
$\neq \begin{bmatrix}0\\0\end{bmatrix}$

line: $\mathbb{R}\text{-span}\left\{v=\begin{bmatrix}v_1\\v_2\end{bmatrix}\right\} = \left\{c\begin{bmatrix}v_1\\v_2\end{bmatrix} \;\middle|\; c\in\mathbb{R}\right\}$



The span of a set of vectors in $\mathbb{R}^3$ is either a point (at the origin), a line (through the origin), a plane (containing the origin), or all of $\mathbb{R}^3$.

plane: $\mathbb{R}\text{-span}\left\{ \begin{array}{l} \text{any two vectors in } \mathbb{R}^2 \\ \text{not on same line: } v \text{ and } w \end{array} \right\} = \mathbb{R}^2$



in this span:

v

w

$v+w$

$v+2w$

$v+3w$

$\vdots$

$2v+w$

$3v+w$

$\vdots$

$av+bw$

2 Multiplying matrices and vectors

We have been using matrices as a compact notation for representing linear systems.

Today we introduce a second way of viewing a matrix: namely, as an operator that transforms one vector to another.

Ex How to multiply an $m \times n$ matrix A with a $n \times 1$ vector $v \in \mathbb{R}^n$

$$\overset{A}{\begin{bmatrix} 1 & 3 & 7 & 0 \\ 1 & 5 & 8 & 1 \\ 2 & 4 & 0 & 0 \end{bmatrix}} \overset{v}{\begin{bmatrix} 2 \\ 1 \\ 4 \\ 3 \end{bmatrix}} \overset{\text{def}}{=} \underbrace{2\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} + 4\begin{bmatrix} 7 \\ 8 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\text{a linear combination of the columns of } A} = \begin{bmatrix} 33 \\ 42 \\ 8 \end{bmatrix}$$

$(m = 3,\ n = 4)$

A is $m \times n$ matrix, $A = [\vec{a}_1 \ \vec{a}_2 \cdots \vec{a}_n]$ each $\vec{a}_i \in \mathbb{R}^m$

$v \in \mathbb{R}^n$ so $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ $\qquad Av \in \mathbb{R}^m$

Definition. If A is a matrix with columns $a_1, a_2, \ldots, a_n \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$, so that

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n \end{bmatrix} \qquad \text{and} \qquad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

then the *matrix-vector product* Av is the vector in $\mathbb{R}^m$ given by:

$$\boxed{Av = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\vec{a}_1 + v_2\vec{a}_2 + \cdots + v_n\vec{a}_n \in \mathbb{R}^m.}$$

Thus Av is the linear combination of the columns of A with coefficients given by the entries of v.

2x3 matrix

3x1 vector

Example. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}$ and $v = \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$ then $a_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, and $a_3 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ so

$$\boxed{Av = 4a_1 + 3a_2 + 7a_3 = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \end{bmatrix} + \begin{bmatrix} -7 \\ 21 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}} \in \mathbb{R}^2$$

3x2

2x1

Example. If $A = \begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix}$ and $v = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ then $a_1 = \begin{bmatrix} 2 \\ 8 \\ -5 \end{bmatrix}$ and $a_2 = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$ so we have

$$\boxed{Av = 4a_1 + 7a_2 = 4\begin{bmatrix} 2 \\ 8 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 8 \\ 32 \\ -20 \end{bmatrix}}_{4a_1} + \underbrace{\begin{bmatrix} -21 \\ 0 \\ 14 \end{bmatrix}}_{7a_2} = \begin{bmatrix} -13 \\ 32 \\ -6 \end{bmatrix}} \in \mathbb{R}^3$$

Av and A have same # of rows

Note: Av is only defined by

$$(\#\text{columns } A) = (\#\text{rows of } v)$$

* If A is $m \times n$ then Av is only defined for $v \in \mathbb{R}^n$, and in this case $Av \in \mathbb{R}^m$.

Thus A transforms vectors in $\mathbb{R}^n$ to vectors in $\mathbb{R}^m$.

This transformation is *linear*:

1. If A is an $m \times n$ matrix and $u, v \in \mathbb{R}^n$ then $\boxed{A(u+v) = Au + Av}$.
2. If A is an $m \times n$ matrix and $v \in \mathbb{R}^n$ and $c \in \mathbb{R}$ then $\boxed{A(cv) = c(Av)}$.

Pf if $A = [\vec{a}_1 \ \vec{a}_2]$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

then $A(v+w) = A\begin{bmatrix} v_1+w_1 \\ v_2+w_2 \end{bmatrix} = (v_1+w_1)\vec{a}_1 + (v_2+w_2)\vec{a}_2$

$= \underbrace{(v_1\vec{a}_1 + v_2\vec{a}_2)}_{=Av} + \underbrace{(w_1\vec{a}_1 + w_2\vec{a}_2)}_{=Aw} = Av + Aw$

why "linear": multiplication by A transforms lines to lines

a <u>line</u> is the span of one nonzero vector $L = \mathbb{R}\text{-span}[v]$

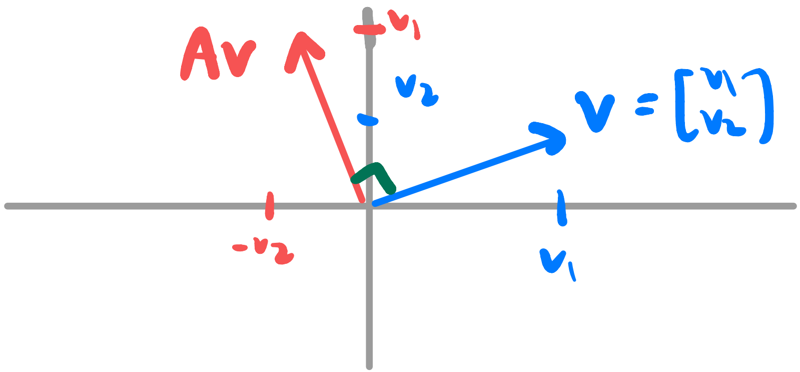
define $AL \overset{\text{def}}{=} \{Aw \mid w \in L\} = \underbrace{\mathbb{R}\text{-span}[Av]}_{\text{another line}}$

Often we can interpret multiplication by 2×2 matrices geometrically

Ex If $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ then

$$A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + v_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ v_1 \end{bmatrix} + \begin{bmatrix} -v_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix}$$

this is just $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ rotated ccw by 90^0



(entry in row k of Av) = (row k of A) x v

Let A and v be the general $m \times n$ matrix and n-row vector given by

n entries

n entries

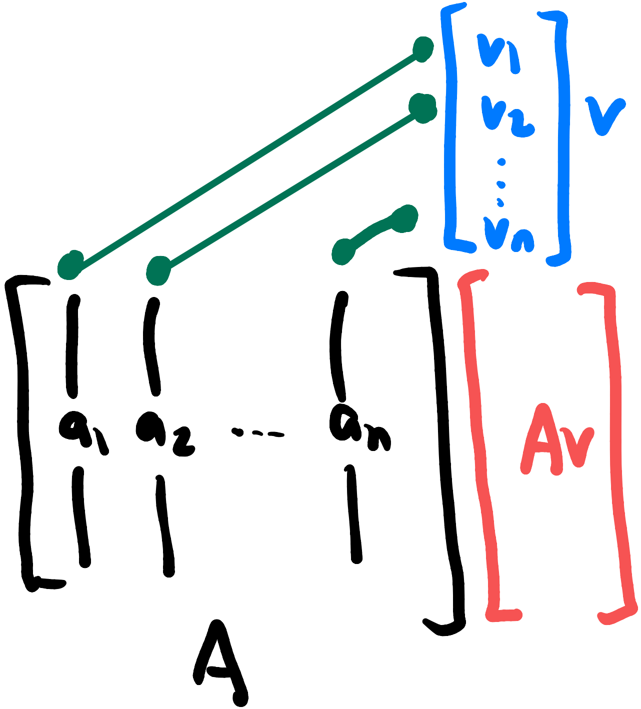
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

To compute Av: match up entries in ith column of A with the ith row of v.

$$Av = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ \boxed{a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n} \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}.$$

For example, $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = 1 \cdot 5 + 2 \cdot 6 + 3 \cdot 7 + 4 \cdot 8 = 5 + 12 + 21 + 32 = 70.$

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$



\# columns of A

=

\# rows of v

\# rows of Av

=

\# rows of A

$A\vec{x} = b$ where A is $m \times n$ matrix of numbers

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a vector of variables

3 Matrix equations

If A is an $m \times n$ matrix with columns $a_1, a_2, \ldots, a_n \in \mathbb{R}^m$ and

$$\boxed{x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} \quad \text{and} \quad \boxed{b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^m}$$

b = vector in $\mathbb{R}^m$ (numeric entries)

where each x_i is a variable, then we call $Ax = b$ a *matrix equation*.

Proposition. The matrix equation $\boxed{Ax = b}$ has the same solutions as both the vector equation $\boxed{x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = b}$ and the linear system whose augmented matrix is $\boxed{\begin{bmatrix} a_1 & a_2 & \ldots & a_n & b \end{bmatrix}}$.

$[A|b]$

Proposition. The matrix equation $Ax = b$ has a solution if and only if b is a linear combination of the columns of A, that is, $\boxed{b \in \mathbb{R}\text{-span}\{a_1, a_2, \ldots, a_n\}}$.

Assume $A = [a_1 \; a_2 \; \cdots \; a_n]$ where $a_i \in \mathbb{R}^m$

linear system

$$\begin{cases} x_1 + 3x_2 - x_3 = 0 \\ x_1 + x_2 + x_3 = 5 \end{cases}$$

augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 1 & 1 & 1 & 5 \end{array}\right]$$

matrix equation

$$\underbrace{\begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix}}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

this is the COEFFICIENT matrix of the system

vector equation

$$x_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

Saying $Ax=b$ has solution for all $b\in\mathbb{R}^3$ $\iff$ Saying b is in span of the columns of A for all $b\in\mathbb{R}^3$ $\iff$ the span of the columns of A is $\mathbb{R}^3$

Example. Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Does $Ax = b$ have a solution for all choices of $b_1, b_2, b_3 \in \mathbb{R}$?

No!

The system $Ax = b$ has a solution if and only if

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix}$$

augmented matrix of a linear system with same solutions as $Ax=b$

is the augmented matrix of a consistent linear system. We can determine if this system is consistent by row reducing the matrix to echelon form:

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ \cancel{-4} & 2 & -6 & b_2 \\ \cancel{-3} & -2 & -7 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & \cancel{7} & 5 & 3b_1 + b_3 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 3 & 4 & b_1 \\ 0 & \boxed{14} & 10 & 4b_1 + b_2 \\ 0 & 0 & 0 & \boxed{b_1 - \frac{1}{2}b_2 + b_3} \end{bmatrix}.$$

do some row ops to cancel entries below (1,1)

in echelon form, where pivots are depends on b

is only a pivot if it is nonzero

there is a choice of $b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ for example that makes last column a pivot

The last matrix is in echelon form, so its leading entries are the pivot positions of our first matrix. The linear system is consistent if and only if the last column does not contain a pivot position. This occurs precisely when $\boxed{b_1 - \frac{1}{2}b_2 + b_3 = 0.}$

But we can choose numbers such that $b_1 - \frac{1}{2}b_2 + b_3 \neq 0$: take $b_1 = 1$ and $b_2 = b_3 = 0$.
Therefore our original matrix equation $\boxed{Ax = b}$ does not always have a solution.

(for some choice of b, we have $b \notin$ span of columns of A)

We can generalize this example:

Theorem. Let A be an $m \times n$ matrix. The following properties are equivalent, meaning that if one of them holds, then they all hold, but if one of them fails to hold, then they all fail:

equivalent props

1. For each vector $b \in \mathbb{R}^m$, the matrix equation $Ax = b$ has a solution.
2. Each vector $b \in \mathbb{R}^m$ is a linear combination of the columns of A.
3. The span of the columns of A is the set $\mathbb{R}^m$.

   (Say this as: "the columns of A span $\mathbb{R}^m$".)
4. A has a pivot position in every row. ← not obviously equivalent, but computable

Pf idea: ① ② ③ correspond to when

$[A|b]$ has no pivot in last column for all b

This is only guaranteed if $\mathrm{RREF}(A) = [\ \ldots\ ]$ has pivots in all rows

If RREF(A) has pivots in all rows then

$$\mathrm{RREF}([A \mid b]) = \left[\mathrm{RREF}(A) \,\middle|\, \begin{matrix} ? \\ ? \\ ? \\ \vdots \end{matrix}\right]$$

$$= \left[\begin{array}{cccc|c} 1 & \cdots & \cdots & \cdots & ? \\ & 1 & \cdots & \cdots & ? \\ & & 1 & \cdots & ? \\ & & & 1 & ? \end{array}\right]$$

no pivots here, no matter what entries appear