MATH 2121 - Lecture #5

Outline:

- Review: matrix eqs
- Linear independence
- Linear transformation

$\overset{A}{(m \times n \text{ matrix})} \times (n \times 1 \text{ vector}) = \underbrace{(m \times 1 \text{ vector})}_{\text{always in span of the columns of the matrix } A}$

1 Last time: multiplying vectors and matrices

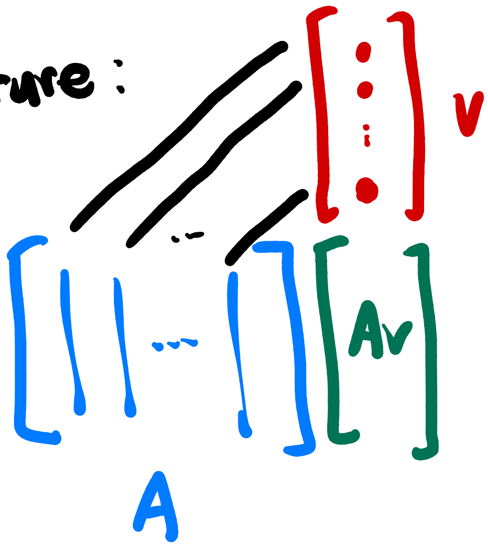
Given a matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ and vector $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ define

$$\boxed{Av = v_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + v_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \in \mathbb{R}^m.}$$

Call Av the product of A and v, or the vector given by multiplying v by A.

Example. $\underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}} \underset{v}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}} = -\begin{bmatrix} 1 \\ 5 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 6 \end{bmatrix} + \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} -1+0+3 \\ -5+0+7 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$

Picture:



Av has same # rows as A

cols of A = # rows of A (if Av is defined)

$\boxed{Ax = b}$

If A is an $m \times n$ matrix, $x = \overbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}^{\text{vector of variables}}$, $\overset{\text{another vector}}{b} \in \mathbb{R}^m$, then $\boxed{Ax = b}$ is a *matrix equation*.

$Ax = b$ has the same solutions as linear system with augmented matrix $\begin{bmatrix} A & b \end{bmatrix}$.

> **Theorem.** Let A be an $m \times n$ matrix. The following are equivalent:
>
> 1. $Ax = b$ has a solution for any $b \in \mathbb{R}^m$.
> 2. The span of the columns of A is all of $\mathbb{R}^m$.
> 3. A has a pivot position in every row.

$$Ax = b$$

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 4 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

same $\longleftrightarrow$ solutions

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 2x_4 = 7 \\ 2x_1 + 4x_2 + 5x_3 + x_4 = 2 \end{cases}$$

Ex. The matrix equation: $Ax=b$

with $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ has

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$$ ← no pivot in this row

so $\begin{cases} \text{for some } b \in \mathbb{R}^3 \\ Ax=b \text{ has no solution} \end{cases}$

(but $Ax=b$ might have solutions for some choices of b but not others) orthogonal to notion of "span"

2 Linear independence

We will now discuss a property of **lists** of vectors that we call *linear independence*.

Let us first explain what this term means for lists of zero, one, two, or three vectors.

(0) The empty list of vectors is always defined to be *linearly independent*. (by convention)

(1) A list containing just one vector $v_1 \in \mathbb{R}^n$ is *linearly independent* if

$$v_1 \neq 0.$$

if the vector is nonzero

(2) A list containing two vectors $v_1, v_2 \in \mathbb{R}^n$ is *linearly independent* if

$$\text{there is no } a \in \mathbb{R} \text{ with } v_1 = av_2 \quad \text{AND} \quad \text{there is no } b \in \mathbb{R} \text{ with } v_2 = bv_1,$$

meaning that **neither vector is a scalar multiple of the other**.

(3) A list containing three vectors $v_1, v_2, v_3 \in \mathbb{R}^n$ is *linearly independent* if

$$v_1 \notin \mathbb{R}\text{-span}\{v_2, v_3\} \quad \text{AND} \quad v_2 \notin \mathbb{R}\text{-span}\{v_1, v_3\} \quad \text{AND} \quad v_3 \notin \mathbb{R}\text{-span}\{v_1, v_2\}.$$

no one vector is in span of the others

no one vector is in span of the others

$\boxed{\text{Definition.}}$

Definition. A list of p vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ is *linearly independent* if

$$v_i \notin \mathbb{R}\text{-span}\{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_p\} \text{ for all } i = 1, 2, 3, \dots, p.$$

When $p = 1$ we interpret "$\mathbb{R}$-span$\{\}$" to mean the set $\{0\}$ with $0 \in \mathbb{R}^n$.

Example. It is easy to tell if lists of one or two vectors are linearly independent.

(a) The single vector $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$ is linearly independent, but $\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$ is not. not linearly independent

(b) $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\right\}$ are linearly independent, but $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\8\\12\end{bmatrix}\right\}$ are not. not linearly independent

not scalar multiples!

$4\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}4\\8\\12\end{bmatrix} \neq \begin{bmatrix}4\\5\\6\end{bmatrix}$

$x\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}x\\2x\\3x\end{bmatrix} \neq \begin{bmatrix}4\\5\\6\end{bmatrix}$ if $x \neq 4$

$\begin{bmatrix}4\\8\\12\end{bmatrix} = 4\begin{bmatrix}1\\2\\3\end{bmatrix}$

If we can find a way to write v_i as a linear comb. of $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_p$ $\Rightarrow$ not linearly independent (but can be hard to find this)

(c) What about the three vectors $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$? Since

$$v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 2v_1 + v_3 \in \mathbb{R}\text{-span}\{v_1, v_3\}$$

the list v_1, v_2, v_3 is **not** linearly independent.

It is not clear how we could prove that three vectors are linearly independent.

For this, it is useful to reformulate our definition of linear independence.

iff vector eq. has ONLY one solution (x=0)

Proposition. A list $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$ is linearly independent if and only if

vector eq:
$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0 \qquad (**)$$
always has solution

has no other solutions besides the trivial solution $\boxed{x_1 = x_2 = \cdots = x_p = 0.}$

A nontrivial solution to (**) is called a *linear dependence* among the vectors $v_1, v_2, \ldots, v_p$.

Proof. Suppose our vectors are **not** linearly independent, so that we can write

$$v_i = a_1 v_1 + \cdots + a_{i-1} v_{i-1} + a_{i+1} v_{i+1} + \cdots + a_p v_p \in \mathbb{R}\text{-span}\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_p\}$$

for some index $i \in \{1, 2, \ldots, p\}$ and some scalars $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_p \in \mathbb{R}$. Then

$$(x_1, \ldots, x_{i-1}, \quad x_i, \quad x_{i+1}, \ldots, x_p) = (a_1, \ldots, a_{i-1}, \quad -1, \quad , a_{i+1}, \ldots, a_p)$$

nontrivial solution to vec eq.

is a nontrivial solution to (**), since even if the a's are all zero, $x_i = -1$ is nonzero.

Conversely, suppose (**) has a nontrivial solution $(c_1, c_2, \ldots, c_p)$.

Then one of the numbers c_i in this solution must be nonzero. But then

$$0 = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

and we can divide both sides by the nonzero number c_i to get

$$0 = \frac{1}{c_i}(c_1 v_1 + c_2 v_2 + \cdots + c_p v_p) = \frac{c_1}{c_i} v_1 + \cdots + \frac{c_{i-1}}{c_i} v_{i-1} + \boxed{v_i} + \frac{c_{i+1}}{c_i} v_{i+1} + \cdots + \frac{c_p}{c_i} v_p.$$

move to other side

This means that

$$v_i = -\frac{c_1}{c_i} v_1 - \cdots - \frac{c_{i-1}}{c_i} v_{i-1} - \frac{c_{i+1}}{c_i} v_{i+1} - \cdots - \frac{c_p}{c_i} v_p \in \mathbb{R}\text{-span}\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_p\}$$

so the given list of vectors is **not** linearly independent. □

Next: how to detect linear independence

Corollary. The columns of a matrix A are linearly independent if and only if A has a pivot position in every column.

Proof. Suppose the columns of A are $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$.

By the previous theorem, these vectors are linearly independent if and only if the matrix equation $\boxed{Ax = 0}$ has exactly one solution (the trivial solution $x = 0$).

This happens precisely when A has a pivot position in every column. □

any non-pivot column gives a free var, making $Ax=b$ have ∞-many solutions

algorithm

How to determine if $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$ are linearly independent.

- Form the $n \times p$ matrix $A = \begin{bmatrix} v_1 & v_2 & \ldots & v_p \end{bmatrix}$.
- Reduce A to echelon form to find its pivot columns.
- If every column of A has a pivot, the vectors are linearly independent.

Example. The vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \text{ and } \begin{bmatrix} 5 \\ 9 \\ 15 \end{bmatrix} \right\}$ are linearly independent, since

①

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 9 \\ -1 & 5 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 9 \\ 0 & 7 & 20 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix} = \mathsf{RREF}(A)$$

②

so every column of A contains a pivot position.

pivots in every column ✓

Columns of A span $\mathbb{R}^m$ $\iff$ A has pivots in every row

Columns of A are linearly independent $\iff$ A has pivots in every column

NOT linearly independent

3 Linear dependence

A list $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$ is *linearly dependent* if it is not linearly independent.

(1) A list containing just one vector $v_1 \in \mathbb{R}^n$ is *linearly dependent* if $v_1 = 0$.

(2) A list containing two vectors $v_1, v_2 \in \mathbb{R}^n$ is *linearly dependent* if

$$\text{there is some } a \in \mathbb{R} \text{ with } v_1 = av_2 \quad \text{OR} \quad \text{there is some } b \in \mathbb{R} \text{ with } v_2 = bv_1,$$

meaning that **one vector is a scalar multiple of the other**.

(3) An arbitrary list $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$ is *linearly dependent* if some

$$v_i \in \mathbb{R}\text{-span}\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_p\}.$$

one vector is in span of the others

This occurs when a linear dependence exists among the vectors, or when

$$\begin{bmatrix} v_1 & v_2 & \ldots & v_p \end{bmatrix}$$

has at least one non-pivot column.

no pivot in this column

Example. The vectors $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$, and $\begin{bmatrix} 5 \\ 9 \\ 16 \end{bmatrix}$ are linearly dependent since

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 9 \\ -1 & 5 & 16 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 9 \\ 0 & 7 & 21 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 3 \\ 0 & 0 & 0 \end{bmatrix} = \mathsf{RREF}(A).$$

more vectors than rows

short cuts

Proposition. If $p > n$ then $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ are linearly dependent.

Proof. If $p > n$ then $\begin{bmatrix} v_1 & v_2 & \dots & v_p \end{bmatrix}$ cannot have a pivot in every column. □

\# pivots ≤ min(#rows, #columns)

Proposition. If any $v_i = 0$ then $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ are linearly dependent.

Proof. If $v_i = 0$ then the matrix $\begin{bmatrix} v_1 & v_2 & \dots & v_p \end{bmatrix}$ has no pivot in column i. □

$$\mathrm{RREF}\left(\begin{bmatrix} ? & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} & ? \end{bmatrix}\right) = \begin{bmatrix} ? & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} & ? \end{bmatrix}$$

has no pivot

first need to define functions

4 Linear transformations

A *function* f takes an input x from some set X and produces an output $f(x)$.

We write $\boxed{f : X \to Y}$ to mean that f is a function that takes inputs from the set X and gives outputs that are contained in the set Y.

3 things

1 The set X is called the *domain* of the function f.

2 The set Y is called the *codomain* of f.

3 a formula that transforms $x \in X$ to an output $f(x) \in Y$

Every element $x \in X$ is a valid input to f.

Not every $y \in Y$ needs to occur as an output of f.

Ex the formula $f(x) = x^2$ defines a function $f : \mathbb{R} \to \mathbb{R}$

but no negative number occurs as $f(x)$ for $x \in \mathbb{R}$

We are interested in functions with vector inputs in $\mathbb{R}^n$ and vector outputs in $\mathbb{R}^m$: $f: \mathbb{R}^n \to \mathbb{R}^m$

Definition. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function whose domain and codomain are sets of vectors. The function f is a $\boxed{\textit{linear transformation}}$ if both of these properties hold:

(1) $f(u + v) = f(u) + f(v)$ for all vectors $u, v \in \mathbb{R}^n$.

(2) $f(cv) = cf(v)$ for all vectors $v \in \mathbb{R}^n$ and scalars $c \in \mathbb{R}$.

} f is "compatible" with vector operations

Example. If A is an $m \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^m$ is the function with the formula $\boxed{T(v) = Av}$ for $v \in \mathbb{R}^n$ then T is a linear function.

Proposition. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then we also have

(3) $f(0) = 0$. as $f(0) = f(0v) \underset{(2)}{=} 0f(v) = 0$

(4) $f(u - v) = f(u) - f(v)$ for $u, v \in \mathbb{R}^n$.

(5) $\boxed{f(a_1v_1 + a_2v_2 + \cdots + a_pv_p) = a_1f(v_1) + \cdots + a_pf(v_p)}$ for any $a_i \in \mathbb{R}$ and $v_i \in \mathbb{R}^n$.

"compatible with linear combinations"

Geometric intuition: if $S \subseteq \mathbb{R}^n$ is any shape

then $f(S) \stackrel{\text{def}}{=} \{ f(v) \mid v \in S \}$ is another

(possibly weird) shape in codomain $\mathbb{R}^m$

But if $f: \mathbb{R}^n \to \mathbb{R}^m$ is linear

and $S = $ (a line in $\mathbb{R}^n$) ← means "the span of one vector"

then $f(S) = $ (a line in $\mathbb{R}^m$)

elementary (basis) vectors

Define $\boxed{e_1, e_2, \ldots, e_n \in \mathbb{R}^n}$ as the vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad e_{n-1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^n$$

Fact. If A is an $m \times n$ matrix then Ae_i is the ith column of A.

For example $\begin{bmatrix} 1 & 2 & \boxed{\begin{matrix}3\\7\end{matrix}} & 4 \\ 5 & 6 & & 8 \end{bmatrix} e_3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$

proof by example

$$= \cancel{0\begin{bmatrix} 1 \\ 5 \end{bmatrix}} + \cancel{0\begin{bmatrix} 2 \\ 6 \end{bmatrix}} + 1\begin{bmatrix} 3 \\ 7 \end{bmatrix} + \cancel{0\begin{bmatrix} 4 \\ 8 \end{bmatrix}}$$

Fundamental thm about linear transformations

Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.

1 Then there is a unique $m \times n$ matrix A such that $\boxed{T(v) = Av}$ for all $v \in \mathbb{R}^n$.

2 The matrix A has the formula $\boxed{A = \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{bmatrix}}$.

Proof. Define $A = \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{bmatrix}$. If $w \in \mathbb{R}^n$ is any vector then

$$T(w) = T\left(\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}\right) = T\left(w_1 \overset{e_1}{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}} + w_2 \overset{e_2}{\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}} + \cdots + w_n \overset{e_n}{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}}\right)$$

S $\to$ $= w_1 T(e_1) + \cdots + w_n T(e_n)$ $\leftarrow$ $= Aw$ by def.

$= Aw.$

Thus A is one matrix such that $T(v) = Av$ for all vectors $v \in \mathbb{R}^n$.

T is a function

T(v) is a vector

[T(e₁) T(e₂) ... T(eₙ)] is a matrix

then

Suppose $T(x) = Ax = Bx$ for all $x \in \mathbb{R}^n$:

The theorem says that A is the the only matrix with this property.

To show this, suppose B is a $m \times n$ matrix with $T(v) = Bv$ for all $v \in \mathbb{R}^n$.

Then $\boxed{T(e_i) = Ae_i = Be_i \text{ for all } i = 1, 2, \ldots, n.}$

But Ae_i and Be_i are the ith columns of A and B.

Therefore A and B have the same columns, so they are the same matrix: $A = B$. $\square$

We call $\boxed{A \text{ the } \textit{standard matrix}}$ of the linear transformation T.

The standard matrix A is the unique matrix such that $T(v) = Av$ for all v.

Example. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is the function $\boxed{T(v) = 3v.}$

This is a linear transformation, and its standard matrix has the formula

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{bmatrix} = \begin{bmatrix} 3e_1 & 3e_2 & \dots & 3e_n \end{bmatrix} = \begin{bmatrix} 3 & 0 & \dots & 0 \\ 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 3 \end{bmatrix}.$$

A has nonzero entries only in positions $(1, 1), (2, 2), \dots, (n, n)$.

One calls such a matrix $\boxed{\textit{diagonal}.}$

Example. Suppose $T : \mathbb{R}^n \to \mathbb{R}$ is the function

$$\boxed{T\left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1^2 + v_2^2 + \cdots + v_n^2.}$$

This function is **not** linear: we have $T(2v) = 4T(v) \neq 2T(v)$ for any $0 \neq v \in \mathbb{R}^n$.

$$T(2v) = (2v_1)^2 + \dots + (2v_n)^2$$

$$= 4(v_1^2 + \dots + v_n^2) = 4T(v) \neq 2T(v)$$

Example. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is the function

$$\boxed{T\left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) = \begin{bmatrix} v_n \\ \vdots \\ v_2 \\ v_1 \end{bmatrix}.}$$

flip input vector

This function is a linear transformation. (Why?) Its standard matrix is

↳ check ① + ②

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_{n-1}) & T(e_n) \end{bmatrix} = \begin{bmatrix} e_n & e_{n-1} & \dots & e_2 & e_1 \end{bmatrix} = \begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix}.$$

In the matrix on the right, all blank space indicates zero entries that are not shown.