MATH 2121 - Lecture #6

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1. Review - linear transformations
2. one-to-one and onto functions
3. how to detect 1 & 2

general idea: $f : X \to Y$ means a function f that takes inputs $a \in X$ (domain) and produces outputs $f(a) \in Y$ (codomain)

1 Last time: linear transformations = linear functions

Writing $\boxed{f : X \to Y}$ means that X is the *domain* (the set of inputs), Y is the *codomain* (a set that contains all outputs and possibly more elements), and f is a function that transforms each input in X to an output that belongs to Y.

Let m and n be positive integers. Recall that $\mathbb{R}^n$ is the set of vectors with n rows.

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a function (whose domain and codomain are sets of vectors).

The following mean the same thing:

- T is *linear*.
- We have $\boxed{T(u+v) = T(u) + T(v)}$ and $\boxed{T(cv) = cT(v)}$ for $u, v \in \mathbb{R}^n$, $c \in \mathbb{R}$. general def.
- There is an $m \times n$ matrix A such that T has the formula $\boxed{T(v) = Av \text{ for } v \in \mathbb{R}^n}$.

Simple examples

What are the linear transformations $f: \mathbb{R} \to \mathbb{R}$? (when $m=n=1$)

① Last item on previous slide says

$f(x) = Ax$ for some 1×1 matrix A

Such a matrix is just a number $A \in \mathbb{R}$

② NOTE: functions of form $\underbrace{f(x) = ax+b}_{\text{"affine" function}}$ $(a,b \in \mathbb{R})$

are not considered to be linear if $b \neq 0$

(even though its graph is a line)

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear. How to find a matrix A with $T(x) = Ax$?

If we are given a linear transformation T, then $T(v) = Av$ for the matrix

A is m×n, given by $\boxed{A = \left[\; T(e_1) \quad T(e_2) \quad \ldots \quad T(e_n) \;\right]}$

where $\boxed{e_i \in \mathbb{R}^n}$ is the vector with a 1 in row i and 0 in all other rows.

We call A the *standard matrix* of T.

Two different linear functions $\mathbb{R}^n \to \mathbb{R}^m$ cannot have the same standard matrix.

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \in \mathbb{R}^n = \text{domain of } T \\ \\ \\ \leftarrow \text{row } i \\ \\ \\ \\ \end{array}$$

Comment Suppose T is linear and you know some values $T(v_1), T(v_2), \dots, T(v_n)$ where $v_1, v_2, \dots, v_n \in \mathbb{R}^n$ are not the vectors $e_1, e_2, \dots, e_n$. How to find $A = [T(e_1) \cdots T(e_n)]$?

Answer: Need to find a way to express each

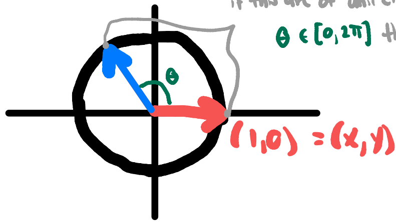
$$e_i = c_1 v_1 + c_2 v_2 + \dots + c_n v_n \quad (c_1, c_2, \dots, c_n \in \mathbb{R})$$

Then can compute $\boxed{T(e_i) = c_1 T(v_1) + c_2 T(v_2) + \dots + c_n T(v_n)}$

holds by def of linearity

Trigonometry review:

Recall: the unit circle is the circle of radius 1 centered at $(x,y) = (0,0)$, has circumference 2π



if this arc of unit circle has length $\theta \in [0, 2\pi]$ then the blue vector to be $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

Example. Fix $\theta \in [0, 2\pi)$. Here $[a, b)$ means "all numbers $x \in \mathbb{R}$ with $a \le x < b$."

Define

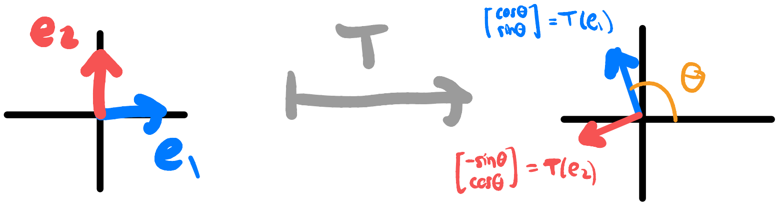
$$\boxed{A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{rotation matrix}}$$

and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $T(v) = Av$.

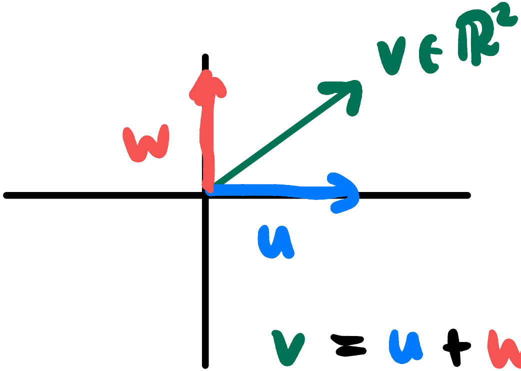
- If $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e_1$ (parallel to the x-axis), then $T(v) = Av = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$.
- If $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e_2$ (parallel to the y-axis), then $T(v) = Av = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta + \frac{\pi}{2}) \\ \sin(\theta + \frac{\pi}{2}) \end{bmatrix}$.

$\frac{1}{4}$ of total circumference

In general, $\boxed{T(v) = Av \text{ is obtained by rotating } v \text{ counterclockwise by the angle } \theta.}$

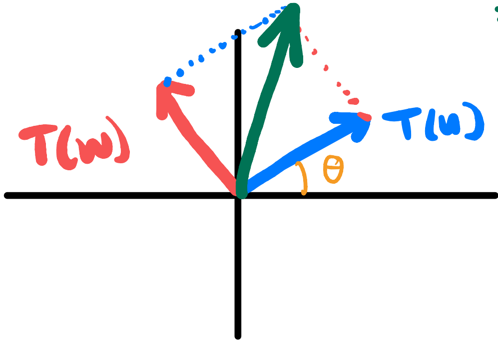


picture to see why T always rotates: $T(u)+T(w) = T(u+w)$
$= T(v)$



$$v = u + w$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



inputs to T

outputs of T

Since $\underbrace{\text{sides}}_{\text{gives us } T(u),\, T(w)}$ are rotated, so is $\underbrace{\text{diagonal}}_{\text{sum of sides which is } T(v) \text{ by linearity}}$

(injective) (surjective)

2 One-to-one and onto functions

Below we introduce two important classes of linear transformations, characterized in terms of whether the standard matrix has pivots in every row or in every column.

To prepare for this, we first discuss some properties of functions in general.

A function has **three components**: a choice of ① **domain**, a choice of ② **codomain**, and a **formula/rule/algorithm** to assign an output in the codomain to each input in the ③ domain. Two functions are the same only when all three are equal.

- The formula $x \mapsto \sqrt{x}$ defines a function $\boxed{f : \mathbb{R}_{\geq 0} \to \mathbb{R}}$ with $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$.
- The formula $x \mapsto \sqrt{x}$ also defines a function $\boxed{g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}}$.

We consider f and g to be different functions because they have different codomains.

even though they have same formula

Notation: often express formulas as "$x \mapsto$ (some expression)"
(not a function yet, need X & Y)

$f : X \to Y$ is some function

For every x in the domain X of f, we get an output $f(x)$.

Some values y in the codomain Y may never occur as outputs of f.

means $f(x)$

The $\boxed{\textit{image} \text{ of an input } x \text{ in } X \text{ under } f}$ is the output $f(x)$.

The *range* of the function f (sometimes called the *image* of f) is the set

$$\mathsf{range}(f) = \{f(x) : x \in X\} \subset \text{codomain}$$

of images of all inputs in the domain.

The range of f is the subset of the codomain Y which gives all actual outputs of f.

smallest possible codomain that could be assigned to given domain & formula to get a function linear

Example. Suppose $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$ and $T : \mathbb{R}^2 \to \mathbb{R}^3$ is the function $\boxed{T(v) = Av.}$

(a) The image of a vector $v \in \mathbb{R}^2$ under T is by definition $\boxed{T(v) = Av.}$

The image of $v = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ under T is

$$T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}.$$

(NO)

(b) $\boxed{\text{Is the range of } T \text{ all of } \mathbb{R}^3\text{?}}$ If it was, then $Ax = b$ would have a solution for all b $b \in \mathbb{R}^3$. But this only holds when A has a pivot position in every row. This is impossible since each column can only contain one pivot position, but A has three rows and only two columns. Therefore $\mathsf{range}(T) \neq \mathbb{R}^3$.

$$\text{range}(T) = \{T(x) \mid x \in \mathbb{R}^2\} = \{Ax \mid x \in \mathbb{R}^2\}$$

$$= (\text{span of columns of } A)$$

whether f is one-to-one depends on domain and formula but not on the codomain.

Definition. A function $f : X \to Y$ is *one-to-one* if $f(a) = f(b)$ implies $a = b$.

Sometimes the term "one-to-one" is called *injective*.

This means that f does not send two different inputs to the same output.

The function f is **not** one-to-one if there are different inputs $a \neq b$ with $f(a) = f(b)$.

Example. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is the linear transformation $T(v) = Av$ where

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 5 & 3 \end{bmatrix}$$

so $Ax = 0$ has infinitely many solutions.

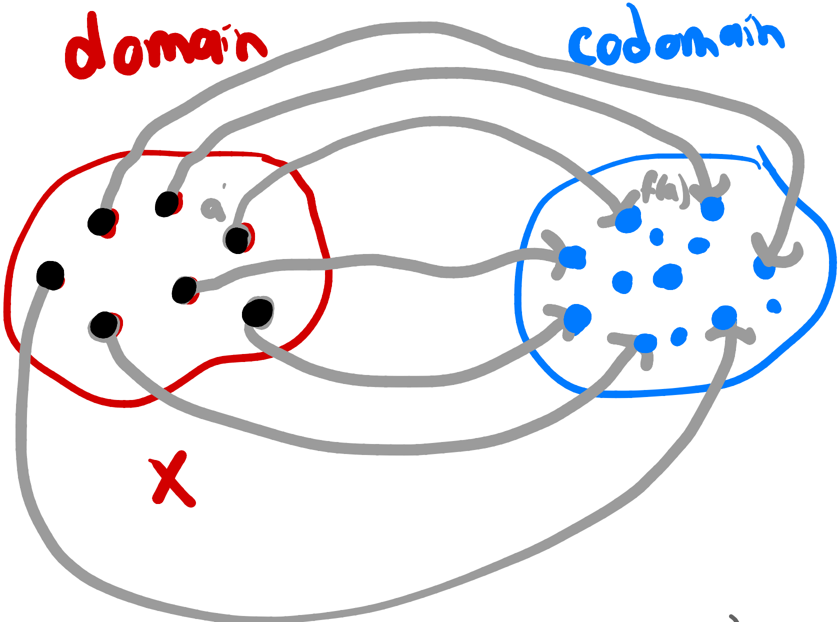
Is T one-to-one? No: since A has more columns than rows, its columns are linearly dependent. Therefore there is a vector $0 \neq v \in \mathbb{R}^3$ such that $T(v) = Av = 0$.

But we also have $T(0) = 0$.

any two solutions $v \neq w$ have $T(v) = T(w) = 0$

Does T send different inputs $v \neq w \in \mathbb{R}^3$ to different outputs $T(v) \neq T(w) \in \mathbb{R}^2$?

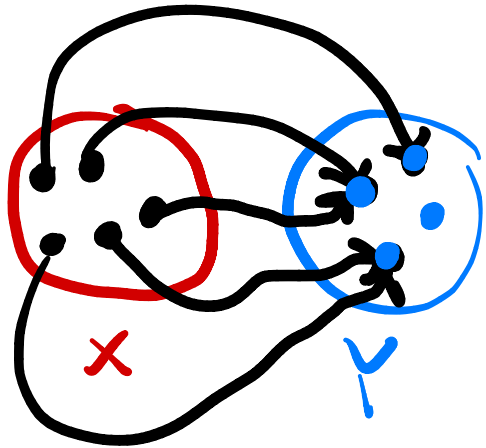
Picture of one-to-one function $f: X \to Y$



(no two arrows share an endpoint)

NOT one-to-one

$f: X \to Y$



(at least two arrow do share endpoint)

Every day examples

① $f : \{\text{students in this class}\} \to \{\text{integers}\}$

$f(\text{student}) = \text{student id}$

one-to-one? $\boxed{\text{YES}}$

② $g : \{\text{students}\} \to \{\text{strings}\}$

$g(\text{student}) = \text{surname}$

one-to-one? usually $\boxed{\text{No}}$

domain $\mathbb{R}^n$
codomain $\mathbb{R}^m$

$(m \times n)$

Theorem. Suppose $\boxed{T : \mathbb{R}^n \to \mathbb{R}^m}$ is a linear function with standard matrix A.

Then the following are equivalent:

(a) T is one-to-one.

(b) The only solution to $T(x) = 0$ is $x = 0 \in \mathbb{R}^n$. $\Leftrightarrow$ only sol. to $Ax=0$ is $x=0$

(c) The columns of A are linearly independent.

(d) Every column of A contains a pivot position. $\leftarrow$ computable

Why are (a) and (b) equivalent? (Uses linearity)

(a) $\Rightarrow$ (b): if T is one to one and $x \neq 0$ then $T(x) \neq T(0) = 0$

(b) $\Rightarrow$ (a): if $x \neq y \in \mathbb{R}^n$ then $x - y \neq 0$ so $T(x) - T(y) = T(x-y) \neq 0$
so $T(x) \neq T(y)$

by linearity

Note: whether a function is onto does depend on choice of codomain.

"range equals codomain"

Definition. A function $f : X \to Y$ is *onto* if

$$\mathsf{range}(f) = \{f(x) : x \in X\} = Y.$$

Sometimes the term "onto" is called *surjective*.

The function f is **not** onto if there is some $y \in Y$ with $f(x) \neq y$ for all $x \in X$.

Example. Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is the linear transformation $T(v) = Av$ where

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 5 & 3 \end{bmatrix}.$$

YES

Is T onto? Yes: for any $b \in \mathbb{R}^2$ there is some $x \in \mathbb{R}^3$ with $T(x) = Ax = b$ since

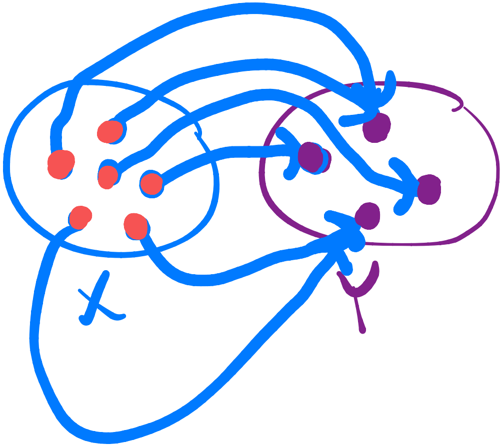
$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 5 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3/5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 19/5 \\ 0 & 1 & 3/5 \end{bmatrix} = \mathsf{RREF}(A),$$

which shows that A has a pivot position in every row.

is $\mathbb{R}^2$ = codomain = range(T) = $\{T(x) \mid x \in \mathbb{R}^3\}$ = $\{Ax \mid x \in \mathbb{R}^3\}$

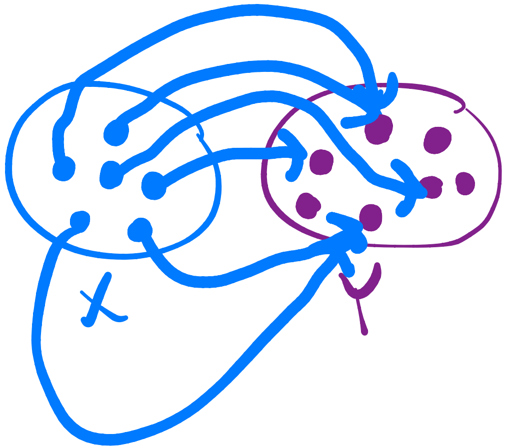
= (span of columns of A) ?

picture of onto $f : X \to Y$



every dot in codomain is endpoint of an arrow

picture of NOT onto $f : X \to Y$



Every day example:

Consider the function

$$f : \{\text{students in this class}\} \to \{A^+, A, A^-, B^+, B, B^-, C^+, C, C^-, D, F\}$$

$$f(\text{student}) = \text{final grade}$$

Is this onto?

large classes: probably $\boxed{\text{YES}}$

small classes: probably $\boxed{\text{No}}$

$(m \times n)$

Theorem. Suppose $\boxed{T : \mathbb{R}^n \to \mathbb{R}^m}$ is linear transformation with standard matrix A.

The following properties are all equivalent:

(a) T is onto. $\Longleftrightarrow$ range(T) = (span of columns of A) = codomain = $\mathbb{R}^m$

(b) The matrix equation $Ax = b$ has a solution for each $b \in \mathbb{R}^m$

(c) The span of the columns of A is $\mathbb{R}^m$.

(d) Every row of A contains a pivot position. $\leftarrow$ computable

Summary: for linear transformations, we can detect one-to-one and onto properties by examining pivot positions of standard matrix.

linear function → one to one? YES

→ onto? NO

Example. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is the function $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} 3v_1 + v_2 \\ 5v_1 + 7v_2 \\ v_1 + 3v_2 \end{bmatrix}$.

This function is a linear transformation. Its standard matrix is $A = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} = \left[T\begin{pmatrix} 1 \\ 0 \end{pmatrix} \; T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$

To determine if T is one-to-one, we check if A has a pivot position in every column. To do this, we convert A to its reduced echelon form:

$$\rightarrow \quad A = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 5 & 7 \\ 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & -8 \\ 0 & -8 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \\ 0 & 0 \end{bmatrix} = \mathsf{RREF}(A).$$

Evidently A does have a pivot position in every column, so T is one-to-one.

As the third row of A does not contain a pivot position, the function T is not onto.

T can only be one-to-one if domain $\mathbb{R}^n$ is "smaller than" codomain $\mathbb{R}^m$ (so $n \leq m$)

Corollary. Consider a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$. $\boxed{T \text{ is one-to-one only if } n \leq m}$, and $\boxed{T \text{ is onto only if } n \geq m}$.

Proof. Let A be the standard matrix of T.

We know that T is one-to-one if and only if A has a pivot position in every column, while T is onto if and only if A has a pivot position in every row.

Each row and each column contains **at most one pivot position**.

Thus if A has a pivot in every column then the number of columns n cannot be more than the number of rows m. Likewise, if A has a pivot in every row then the number of columns m cannot be more than the number of columns n.

This means that if T is one-to-one then $n \leq m$ and if T is onto then $m \leq n$. □

T can only be onto if codomain $\mathbb{R}^m$ is "smaller than" domain $\mathbb{R}^n$ (so $m \leq n$)