

MATH 2121 - Lecture #14

* Next week: off line + online HW

Outline:

- ① Review + more on abstract vector spaces
- ② Eigenvalues + eigenvectors

"concrete" vector space: $\mathbb{R}^n$ and its subspaces

(abstract)

1 Last time: vector spaces

A (real) vector space V is a set containing a zero vector, denoted 0 , with vector addition and scalar multiplication operations that let us produce new vectors $u + v \in V$ and $cv \in V$ from given elements $u, v \in V$ and $c \in \mathbb{R}$.

Several conditions must be satisfied so that these operations behave exactly like vector addition and scalar multiplication for $\mathbb{R}^n$. Most importantly, we require that

1. $u + v = v + u$ and $(u + v) + w = u + (v + w)$.
2. $v - v = 0$ where we define $u - v = u + (-1)v$.
3. $v + 0 = v$
4. $cv = v$ if $c = 1$.

There are a few other more conditions to give the full definition (see the notes).

By convention, we refer to elements of vector spaces as *vectors*.



Example. All subspaces of $\mathbb{R}^n$ are vector spaces, with the usual zero vector and vector operations.

Example. The set of $m \times n$ matrices is a vector space, with the usual addition and scalar multiplication operations.

The zero vector in this vector space is the $m \times n$ zero matrix.

"new" example

important source of examples

Common vector spaces are subspaces of $\mathbb{R}^n$ or subspaces of the following:

Proposition. Let X be a set and let V be a vector space. (e.g. $V = \mathbb{R}$)

Then the set $\text{Functions}(X, V)$ of all functions $f : X \rightarrow V$ is a vector space with

$$\begin{aligned} f + g &= (\text{ the function that maps } x \mapsto f(x) + g(x) \text{ for } x \in X), \\ cf &= (\text{ the function that maps } x \mapsto c \cdot f(x) \text{ for } x \in X), \\ 0 &= (\text{ the function that maps } x \mapsto 0 \in V \text{ for } x \in X), \end{aligned} \quad \left. \vphantom{\begin{aligned} f + g \\ cf \\ 0 \end{aligned}} \right\} \text{ pointwise operations}$$

for $f, g \in \text{Functions}(X, V)$ and $c \in \mathbb{R}$.

useful to think through why

$$\mathbb{R}^n \longleftrightarrow \text{Functions}(X, \mathbb{R}) \text{ where } X = \{1, 2, 3, \dots, n\}$$

$$\begin{bmatrix} f(1) \\ \vdots \\ f(n) \end{bmatrix} \longleftrightarrow f$$

lots of relevant vocab but same defs as for subspaces



A subset $H \subseteq V$ is a *subspace* if

$$0 \in H \quad \text{and} \quad u + v \in H \quad \text{and} \quad cv \in H \quad \text{for all } u, v \in H \text{ and } c \in \mathbb{R}.$$

Let V be a vector space. The definitions of linear combinations, span and linear independence for vectors in V are the same as for vectors in $\mathbb{R}^n$.



Remember that we can only evaluate the linear combination $c_1v_1 + c_2v_2 + \dots$ if it is a finite sum, or if there are finitely many nonzero scalars $c_i \neq 0$.

Example. The subspace of polynomials in $\text{Functions}(\mathbb{R}, \mathbb{R})$ is the span of $1, x, x^2, x^3, \dots$.

But $e^x = 1 + x + \frac{1}{2}x + \frac{1}{6}x^2 + \frac{1}{24}x^3 + \dots + \frac{1}{n!}x^n + \dots$ is not in this subspace.

Definition. A basis of a vector space V is a subset of linearly independent vectors whose span is V .

{ Saying $b_1, b_2, b_3, \dots$ is a basis for V is the same as saying that for each $v \in V$, there are unique coefficients $x_1, x_2, x_3, \dots \in \mathbb{R}$, all but finitely many of which are zero, such that $v = x_1 b_1 + x_2 b_2 + x_3 b_3 + \dots$

{ Theorem. Let V be a vector space. Then V has at least one basis, and every basis of V has the same number of elements (but this could be infinite).

Definition. The dimension of a vector space V is the number $\dim V$ of elements in any of its bases.

Example. If X is a finite set then $\dim \text{Functions}(X, \mathbb{R}) = |X|$.

$E_{\mathbb{R}^{m \times n}}$ = vector space of $m \times n$ matrices

$\dim(\mathbb{R}^{m \times n}) = m \cdot n$ as basis is $\{E_{ij} \mid 1 \leq j \leq n\}$

$\hookrightarrow$ has single nonzero entry, 1 in (i,j)

Consider $m=n=2$ case:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E_{11}} + b \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{E_{12}} + c \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{E_{21}} + d \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_{22}}$$

"LHS"

this shows that $\mathbb{R}^{2 \times 2} = \mathbb{R}\text{-span}\{E_{11}, E_{12}, E_{21}, E_{22}\}$

also these are linearly independent because

the only way to make the LHS = $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

is by taking $a=b=c=d=0$

Claim $\dim(\text{Functions}(X, \mathbb{R})) = |X|$ (the size of X)

Here is a basis of size $|X|$: define for each $y \in X$,

a function $b_y: X \rightarrow \mathbb{R}$

$$b_y(x) = \begin{cases} 1 & \text{if } x=y \\ 0 & \text{if } x \neq y \end{cases}$$

$|X|$ different functions

① Spanning

any $f: X \rightarrow \mathbb{R}$ is equal to $\sum_{y \in X} f(y) b_y$

$\downarrow$
 $\in \mathbb{R}$
 $\uparrow$
function

$$f(x) = \sum_{y \in X} f(y) \underbrace{b_y(x)}_{\substack{=0 \text{ if } y \neq x \\ =1 \text{ if } y=x}} = \left(\sum_{y \in X} f(y) b_y \right)(x) \text{ for all } x \in X.$$

② Independence if $\sum_{y \in X} c_y b_y = (\text{zero function})$ then

$$0 = \left(\sum_{y \in X} c_y b_y \right)(x) = c_x \Rightarrow \text{all coeffs } c_y = 0$$

2 Linear functions

Suppose U and V are vector spaces.

Definition. A function $f : U \rightarrow V$ is *linear* if

$$f(u + v) = f(u) + f(v)$$

and

$$f(cv) = cf(v)$$

for all $c \in \mathbb{R}$ and $u, v \in U$.

Proposition. If U, V, W are vector spaces and $f : V \rightarrow W$ and $g : U \rightarrow V$ are linear functions then $f \circ g : U \rightarrow W$ is also linear.

Recall that we define $f \circ g(x) = f(g(x))$ for $x \in U$.

Proof. The proof is the same as when $U = \mathbb{R}^n$, $V = \mathbb{R}^m$, and $W = \mathbb{R}^p$. $\square$

key difference: no longer have def of standard matrix

Let $\text{Lin}(U, V)$ be the subspace of linear functions $f : U \rightarrow V$ in $\text{Functions}(U, V)$.

Proposition. If V is a finite-dimensional vector space then $\dim \text{Lin}(V, \mathbb{R}) = \dim V$.

Proof. Suppose $b_1, b_2, \dots, b_n$ is a basis for V .

A basis for $\text{Lin}(V, \mathbb{R})$ is given by the linear functions $\phi_1, \phi_2, \dots, \phi_n : V \rightarrow \mathbb{R}$ with

$$\phi_i(x_1 b_1 + x_2 b_2 + \dots + x_n b_n) = x_i \quad \text{for } x_1, x_2, \dots, x_n \in \mathbb{R}.$$

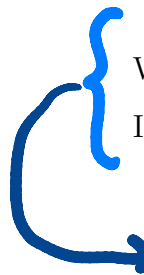
The unique way to express any linear $f : V \rightarrow \mathbb{R}$ as a combination of these is

$$f = f(b_1)\phi_1 + f(b_2)\phi_2 + \dots + f(b_n)\phi_n.$$

□

When $V = \mathbb{R}^n$, we can think of $\text{Lin}(\mathbb{R}^n, \mathbb{R})$ as the vector space of $1 \times n$ matrices.

If $b_1 = e_1, b_2 = e_2, \dots, b_n = e_n$ is the standard basis, then $\phi_i = e_i^\top$.

 When $V = \mathbb{R}^n$, we can think of $\text{Lin}(\mathbb{R}^n, \mathbb{R})$ as the vector space of $1 \times n$ matrices.
If $b_1 = e_1, b_2 = e_2, \dots, b_n = e_n$ is the standard basis, then $\phi_i = e_i^\top$.
a row vector $w = [w_1, \dots, w_n]$
defines a linear function $\mathbb{R}^n \rightarrow \mathbb{R}$ given by $\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \mapsto wv \in \mathbb{R}$

abstract vector spaces:

$$\begin{aligned} \text{Col}(A) &\leftrightarrow \text{range}(f) = \{f(x) \mid x \in U\} \\ \text{Nul}(A) &\leftrightarrow \text{kernel}(f) = \{x \in U \mid f(x) = 0\} \end{aligned}$$

Definition. Suppose U and V are vector spaces and $f : U \rightarrow V$ is a linear function.

Define $\text{range}(f) = \{f(x) : x \in U\} \subseteq V$ and $\text{kernel}(f) = \{x \in U : f(x) = 0\} \subseteq U$.

These subspaces generalize the column space and null space of a matrix.

We have a version of the rank-nullity theorem for arbitrary vector spaces:

* **Theorem** (Rank-Nullity Theorem). If $\dim U < \infty$ then

$$\dim \text{range}(f) + \dim \text{kernel}(f) = \dim U. = \dim(\text{domain})$$

This specializes to our earlier statement about matrices when $U = \mathbb{R}^n$ and $V = \mathbb{R}^m$.

For a self-contained proof, see the lecture notes.

Ex We have discussed a basis for $\mathbb{R}^{3 \times 3}$. What about these subspaces?

A basis for 3×3 symmetric matrices: $(x^T = x)$ $\left\{ \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} : a, b, c, d, e, f \in \mathbb{R} \right\}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

A basis for 3×3 skew-symmetric matrices: $(x^T = -x)$ $\left\{ \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix} \right\}$

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Notice $\dim \mathbb{R}_{\text{sym}}^{3 \times 3} + \dim \mathbb{R}_{\text{skew}}^{3 \times 3} = 6 + 3 = 9$
 $= \dim \mathbb{R}^{3 \times 3}$

This is example of our rank-nullity theorem:

$$f: \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3}$$
$$X \mapsto X + X^T$$

Then $\text{range}(f) = \mathbb{R}_{\text{sym}}^{3 \times 3}$
 $\text{kernel}(f) = \mathbb{R}_{\text{skew}}^{3 \times 3}$

only defined
for square matrices

3 Eigenvectors and eigenvalues

Let A be a square $n \times n$ matrix.

Definition. An *eigenvector* of A is a nonzero vector $v \in \mathbb{R}^n$ such that

$$Av = \lambda v$$

for a number $\lambda \in \mathbb{R}$. (λ is the Greek letter “lambda.”)

The number λ is called the *eigenvalue* of A for the eigenvector v .

Why require nonzero? Because if $v = 0$ then $Av = \lambda v = 0$ for all numbers $\lambda \in \mathbb{R}$.

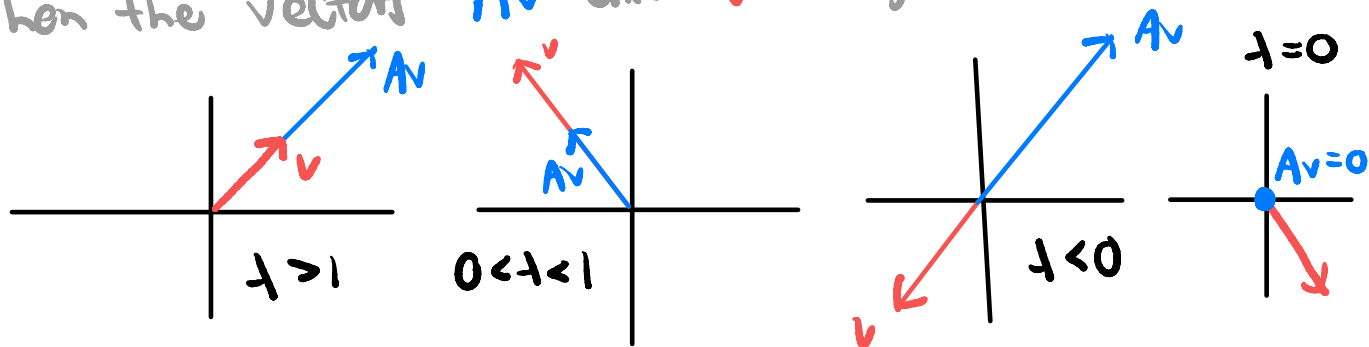
The number 0 is allowed to be an eigenvalue of A , however.

“eigen” $\equiv$ “same”

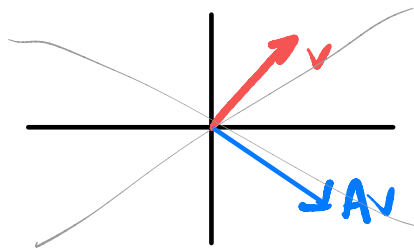
v is an eigenvector for A
when v and Av are both
on same line

Suppose A is a 2×2 matrix and $v \in \mathbb{R}^2$ is one of its eigenvectors
 (so $Av = \lambda v$ for some $\lambda \in \mathbb{R}$)

Then the vectors Av and v might look like this:



but this picture is NOT possible:



v and Av
 are NOT on same
 line!

Ex what are eigenvectors of A with eigenvalue $\lambda=0$?

these are the nonzero vectors v with $Av = 0v = 0$

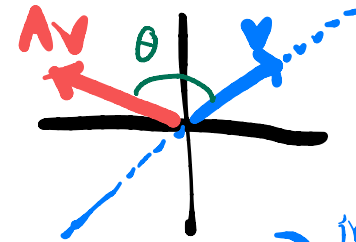
in other words: the nonzero elements of $\text{Nul } A$

Ex what are the eigenvectors of a rotation matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$?

recall that $Av = (v \text{ rotated CCW by } \theta \text{ radians})$

so if θ is not a multiple of π then Av is NOT on same line as v :

$\Rightarrow$ if $\theta \notin \{ \dots, -\pi, 0, \pi, 2\pi, 3\pi, \dots \}$ then
 A has no eigenvectors



But if $\theta=0$ then $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and if $\theta=\pi$ then $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ } in these cases all nonzero vectors are eigenvectors

Example. Given A and v , can check whether v is an eigenvector by computing Av .

If $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ and $v = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ then $Av = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4v$.

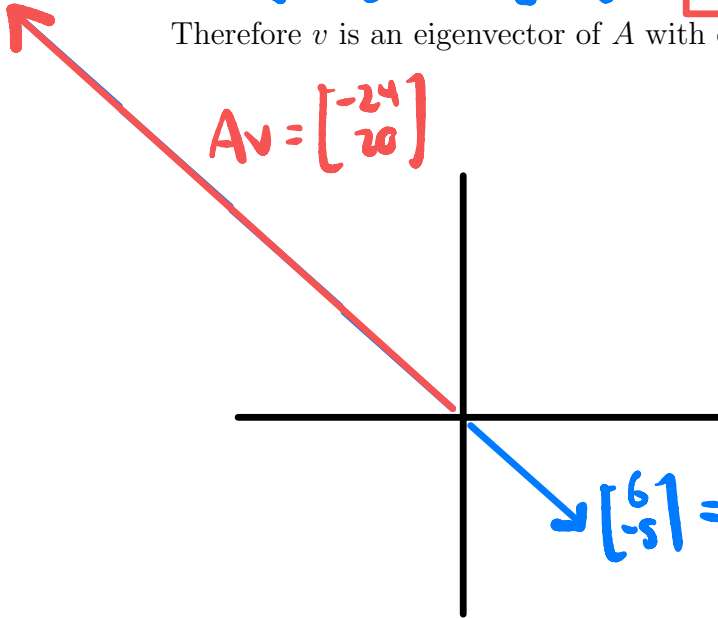
Therefore v is an eigenvector of A with eigenvalue -4 .

$$Av = \begin{bmatrix} -24 \\ 20 \end{bmatrix}$$

so v is eigenvector
for A with eigenvalue

$$\lambda = -4$$

$$\rightarrow \begin{bmatrix} 6 \\ -5 \end{bmatrix} = v$$



Example. What are the eigenvectors of the matrix $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$?

If $v \in \mathbb{R}^4$ were an eigenvector with eigenvalue λ then

$$Av = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_3 \\ v_4 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}.$$

only holds if

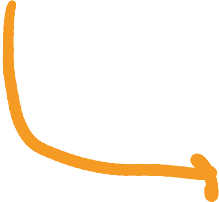
$$\left. \begin{aligned} 0 &= \lambda v_1 \\ v_4 &= \lambda v_2 \\ v_3 &= \lambda v_3 \\ v_2 &= \lambda v_4 \end{aligned} \right\} \begin{aligned} 0 &= 0 \\ v_4 &= 0 \\ v_3 &= 0 \\ v_2 &= 0 \end{aligned}$$

The last equation implies that $0 = \lambda v_4$ and $v_4 = \lambda v_3$ and $v_3 = \lambda v_2$ and $v_2 = \lambda v_1$. So

$$0 = \lambda v_4 = \lambda^2 v_3 = \lambda^3 v_2 = \lambda^4 v_1.$$

If $\lambda \neq 0$ then this would mean that $v_1 = v_2 = v_3 = v_4 = 0$, but remember that v should be nonzero. Therefore the only possible eigenvalue of A is $\lambda = 0$.

The eigenvectors of $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with eigenvalue 0 are


$$v = \begin{bmatrix} x \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{where } x \neq 0.$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} x \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

“ λ is an eigenvalue of A ” means: “there is some $0 \neq v \in \mathbb{R}^n$ such that $Av = \lambda v$.”

Recall that I_n denotes the $n \times n$ identity matrix. We abbreviate by setting $I = I_n$.

Proposition. $\lambda \in \mathbb{R}$ is an eigenvalue of A if and only if $A - \lambda I$ is not invertible.

Proof. $Ax = \lambda x$ has a nonzero solution $x \in \mathbb{R}^n$ if and only if $(A - \lambda I)x = 0$ has a nonzero solution, which occurs if and only if $A - \lambda I$ is not invertible. $\square$

$$(A - \lambda I)x = Ax - \lambda Ix = Ax - \lambda x$$

Example. If $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ then

$$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \text{RREF}(A - 7I).$$

As $\text{RREF}(A - 7I) \neq I$, the matrix $A - 7I$ is not invertible so 7 is an eigenvalue of A .

Corollary. A number $\lambda \in \mathbb{R}$ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$.

Proof. Remember that $A - \lambda I$ is not invertible if and only if $\det(A - \lambda I) = 0$. $\square$

Another way of defining an **eigenvector** the eigenvectors of A with eigenvalue λ are precisely the nonzero elements of the null space $\text{Nul}(A - \lambda I)$.

{ Since we know how to construct a basis for the null space of any matrix, we also know how to find all eigenvectors of a matrix for any given eigenvalue.

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \quad \lambda = 7 \text{ is eigenvalue of } A$$

Example. In the previous example, $\text{RREF}(A - 7I) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so $Ax = 7x$ if and only if $(A - 7I)x = 0$ if and only if $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 - x_2 = 0$. In this linear system, x_2 is a free variable, and we can rewrite x as $x = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

This means $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a basis for $\text{Nul}(A - 7I)$.

So every eigenvector of A with eigenvalue 7 has the form $\begin{bmatrix} a \\ a \end{bmatrix}$ for some $a \in \mathbb{R}$.

$\rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for A
with eigenvalue 7

$$\text{Check: } \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+6 \\ 5+2 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \checkmark$$

$$\text{Nul}(A - \lambda I) = \{x \in \mathbb{R}^n \mid Ax = \lambda x\}$$

One calls the set of all $v \in \mathbb{R}^n$ with $Av = \lambda v$ the *eigenspace* of A for λ . We also call this the λ -*eigenspace* of A . This is just the null space of $A - \lambda I$.

A number is an eigenvalue of A if and only if the corresponding eigenspace is nonzero (that is, contains a nonzero vector).

λ is an eigenvalue
for A iff the λ -eigenspace
of A is
NONZERO

if λ is not an
eigenvalue, then the λ -eigenspace is $\{0\}$.

what is basis for the 2-eigenspace?

Example. Suppose we were told that $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ has 2 as an eigenvalue.

To find a basis for the 2-eigenspace of A , we row reduce

$$A - 2I = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{RREF}(A - 2I).$$

Thus $Ax = 2x$ if and only if $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 - \frac{1}{2}x_2 + 3x_3 = 0$, meaning

linear system

$$x = \begin{bmatrix} \frac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \leadsto \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \text{ is basis for 2-eigenspace}$$

all eigenvectors for A w/ eigval $\neq 2$ have form $a \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$
where $a \neq 0$ or $b \neq 0$



Recall that a matrix is **triangular** if its nonzero entries all appear on or above the main diagonal, or all appear on or below the main diagonal.

Theorem. The eigenvalues of a triangular square matrix A are its diagonal entries.

Proof. If A has diagonal entries $d_1, d_2, \dots, d_n$ then $A - \lambda I$ is triangular with diagonal entries $d_1 - \lambda, d_2 - \lambda, \dots, d_n - \lambda$, so

$$\det(A - \lambda I) = (d_1 - \lambda)(d_2 - \lambda) \cdots (d_n - \lambda)$$

want this to be zero

which is zero if and only if $\lambda \in \{d_1, d_2, \dots, d_n\}$. □



Example. The eigenvalues of the matrix $\begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ are 3, 0, and 2.

Another family of matrices with "easy" determinants:

permutation matrices X

$$\det(X) = (-1)^{\text{inv}(X)}$$

Can we find eigenvalues of permutation matrices X ?

HW: every such X is finite-order:
exists k such that $X^k = I$

Thus if $Xv = \lambda v$ then $X^2v = X(\lambda v) = \lambda Xv = \lambda^2 v$
 $X^3v = \lambda^3 v \dots$

$$\leadsto v = Iv = X^k v = \lambda^k v \text{ so } \boxed{\lambda^k = 1}$$

so if $\lambda \in \mathbb{R}$
is a real
eigenvalue
of X

then

$$\boxed{\lambda = \pm 1}$$