Instructions: Complete the following exercises.

Solutions must be hand-written and submitted in-person.

You will be graded on clarity and simplicity as well as correctness.

You may use any resources and work with other students, but you must write up your own solutions.

Due on **Tuesday, February 24**.

Throughout, let $\mathbb{K}$ be an algebraically closed field and let A be an algebra over $\mathbb{K}$.

1. Show that any nonzero finite-dimensional representation of A has an irreducible subrepresentation.

   (Remember that irreducible representations are required to be nonzero.)

2. Show that the regular representation of $\mathbb{K}[x]$ has no irreducible subrepresentations.

3. Let $Z(A) = \{z \in A : az = za \text{ for all } a \in A\}$. This is called the *center* of A, and it is a subalgebra.

   Suppose (ρ, V) is an irreducible finite-dimensional representation of A. Show that there exists an algebra morphism $\chi : Z(A) \to \mathbb{K}$ such that $\rho(z)(v) = \chi(z)v$ for all $z \in Z(A)$ and $v \in V$.

   The map χ is called the *central character* of (ρ, V).

4. Suppose (ρ, V) is an indecomposable finite-dimensional representation of A.

   Show that if $z \in Z(A)$ then $\rho(z)$ has exactly one eigenvalue $\chi(z) \in \mathbb{K}$ as a linear operator $V \to V$, and that the corresponding map $\chi : Z(A) \to \mathbb{K}$ is again an algebra morphism.

   To prove this, you may use the following results from linear algebra. Suppose $L : V \to V$ is a linear operator. The *generalized eigenspace* of $\lambda \in \mathbb{K}$ is the subspace

   $$V_\lambda = \{v \in V : (L - \lambda)^m v = 0 \text{ for some } m \geq 1\}.$$

   An element $\lambda \in \mathbb{K}$ is called a *generalized eigenvalue* of L if $V_\lambda \neq 0$. It holds that $V = \bigoplus_\lambda V_\lambda$ where the internal direct sum is over the finite set of generalized eigenvalues of L.

5. Define $\rho_b : A \to A$ for $b \in A$ to be map with $\rho_b(a) = ab$ for all $a \in A$.

   Check that $\{\rho_b : b \in A\}$ is a subalgebra of $\mathrm{End}(A)$ isomorphic to A^{op}. Then show that each morphism from the regular representation of A to itself has the form ρ_b for a unique $b \in A$.

   This shows that we have $\mathrm{End}_A(A) \cong A^{\mathsf{op}}$ as algebras, where if V is a left A-module then $\mathrm{End}_A(V)$ denotes the algebra of morphisms of left A-modules $V \to V$.

6. Fix a positive integer N. Suppose $A = \mathbb{K}[x_1, x_2, \dots, x_n]$ and $I \neq A$ is a proper ideal containing all homogeneous polynomials of degree at least N. Show that the quotient A/I of the regular representation is an indecomposable representation of A.

7. Let (ρ, V) be a representation of A. A nonzero vector $v \in V$ is *cyclic* if $V = \{\rho(a)(v) : a \in A\}$.

   Show that (ρ, V) is irreducible if and only if all nonzero vectors in V are cyclic.

8. A left A-module V is *cyclic* if there exists a vector $v \in V$ with $V = \{av : a \in A\}$.

   Show that this occurs if and only if V is isomorphic to A/I for some left ideal in A.

9. Assume $\mathbb{K}$ has characteristic zero.

   Describe the finite-dimensional representations of the *Weyl algebra* $A = \langle x, y : yx - xy = 1 \rangle$.

10. Assume $\mathbb{K} = \mathbb{C}$ and let q be a nonzero complex number.

    Consider the *q-Weyl algebra* $A = \langle x, x^{-1}, y, y^{-1} : xy = qyx \text{ and } xx^{-1} = x^{-1}x = yy^{-1} = y^{-1}y = 1 \rangle$.

    Characterize the values of q such that A has finite-dimensional representations.

    Then describe all finite-dimensional irreducible representations of A for such q.