Instructions: Complete the following exercises.

Solutions must be hand-written and submitted in-person.

You will be graded on clarity and simplicity as well as correctness.

You may use any resources and work with other students, but you must write up your own solutions.

Due on **Tuesday, March 3**.

Throughout, all algebras and vector spaces are defined over an arbitrary field $\mathbb{K}$.

1. Let U, V, and W be vector spaces.

   Let π be the map $V \times W \to V \otimes W$ with $\pi(v, w) = v \otimes w$ for $v \in V$ and $w \in W$.

   (a) Check that π is bilinear.

   (b) Then show that if $f : V \times W \to U$ is any bilinear map, there exists a unique linear map

   $$\tilde{f} : V \otimes W \to U$$

   with $f = \tilde{f} \circ \pi$.

   (c) Prove that $f \mapsto \tilde{f}$ is a linear bijection from the vector space of bilinear maps $V \times W \to U$ to the vector space of linear maps $V \otimes W \to U$.

2. Let V and W be vector spaces.

   Suppose Z is a vector space and $\phi : V \times W \to Z$ is a bilinear map.

   Assume that whenever $g : V \times W \to U$ is a bilinear map to some vector space U, there exists a unique linear map $\tilde{g} : Z \to U$ with $g = \tilde{g} \circ \phi$.

   Show that there is a unique isomorphism $\theta : Z \xrightarrow{\sim} V \otimes W$ such that $\pi = \theta \circ \phi$.

3. Let V and W be vector spaces with bases $\{v_i\}_{i \in I}$ and $\{w_j\}_{j \in J}$.

   Show that $\{v_i \otimes w_j\}_{(i,j) \in I \times J}$ is a basis for $V \otimes W$.

4. Show that $\alpha = \arccos(1/3)/\pi$ is not a rational number.

   The angles between incident faces in a regular tetrahedron are all $\pi\alpha$, while the angles between incident faces in a cube are all $\pi/2$. Using this, check that the Dehn invariant of a regular tetrahedron is nonzero while the Dehn invariant of a cube is always zero. (See Problem 2.13.1 in the textbook.)

   Assume $\mathbb{K}$ is algebraically closed but consider the cases of zero and positive characteristic separately.

5. Show that if $W \subset V$ are finite-dimensional representations of A, then the characters of V, W, and V/W satisfy $\chi_V = \chi_W + \chi_{V/W}$.

6. Suppose $\mathbb{K} = \mathbb{R}$ and A is the algebra of continuous functions $f : \mathbb{R} \to \mathbb{R}$ with $f(x + 1) = f(x)$ for all x. The product for this algebra is point-wise multiplication and the unit element is $f(x) = 1$.

   Let M be the A-module of continuous functions $f : \mathbb{R} \to \mathbb{R}$ with $f(x + 1) = -f(x)$ for all x.

   Show that A and M are indecomposable, non-isomorphic A-modules.

   Show, however, that $A \oplus A \cong M \oplus M$ as A-modules.

7. Show that if m and n are positive integers then $\mathrm{Mat}_m(\mathbb{K}) \otimes \mathrm{Mat}_n(\mathbb{K}) \cong \mathrm{Mat}_{mn}(\mathbb{K})$ as algebras.

8. Assume $\mathbb{K} = \mathbb{C}$ and let V a finite-dimensional complex vector space with a symmetric bilinear form

   $$(\cdot, \cdot) : V \times V \to \mathbb{C}.$$

This form is said to be *nondegenerate* if for each $0 \neq v \in V$ there exists $w \in V$ with $(v, w) \neq 0$.

Show that the following are equivalent:

(a) The form $(\cdot, \cdot)$ is nondegenerate.

(b) For each $v \in V$ the map $v \mapsto (v, \cdot)$ is an isomorphism of vector spaces $V \to V^*$.

(c) If $v_1, v_2, \ldots, v_n$ is a basis for V then the matrix $[(v_i, v_j)]_{1 \leq i,j \leq n}$ is invertible.

9. Continue to assume $\mathbb{K} = \mathbb{C}$ and let V a finite-dimensional complex vector space with basis $v_1, v_2, \ldots, v_n$.

   The *Clifford algebra* $\mathsf{Cliff}(V)$ is the quotient of the tensor algebra TV by the ideal

$$\langle v \otimes v - (v, v)1 : v \in V \rangle.$$

   Let $a_{ij} = (v_i, v_j)$. Then show that $\mathsf{Cliff}(V)$ is isomorphic to the algebra generated by $v_1, v_2, \ldots, v_n$ subject to the relations $v_i v_j + v_j v_i = 2a_{ij}$ and $v_i^2 = a_{ii}$ for all $1 \leq i, j \leq n$ with $i \neq j$.

10. Again let $\mathbb{K} = \mathbb{C}$ and $\dim(V) = n < \infty$. Suppose $(\cdot, \cdot)$ is nondegenerate.

    Show that $\mathsf{Cliff}(V)$ is semisimple and classify its irreducible representations up to isomorphism.