Instructions: Complete the following exercises.

Solutions must be hand-written and submitted in-person.

You will be graded on clarity and simplicity as well as correctness.

You may use any resources and work with other students, but you must write up your own solutions.

Due on **Tuesday, March 10**.

1. Let V a finite-dimensional complex vector space with a symmetric bilinear form
$$(\cdot,\cdot) : V \times V \to \mathbb{C}.$$
Show that $\mathsf{Cliff}(V)$ is not semisimple if $(\cdot,\cdot)$ is degenerate.
What is $\mathsf{Cliff}(V)/\mathsf{Rad}(\mathsf{Cliff}(V))$ when $(\cdot,\cdot)$ is degenerate?

In the following exercise, suppose (V, ρ_V) and (W, ρ_W) are two representations of A.

Here A is an algebra defined over an algebraically closed field $\mathbb{K}$.

Let U be the vector space $V \oplus W = \{(v, w) : v \in V,\ w \in W\}$

Suppose $f : A \to \mathrm{Hom}_{\mathbb{K}}(W, V)$ is a linear map. Define $\rho_f : A \to \mathrm{End}(U)$ to be the map with
$$\rho_f(a)(v, w) = (\rho_V(a)(v) + f(a)(w), \rho_W(a)(w))) \quad \text{for } a \in A,\ v \in V,\ w \in W.$$

2. Find a necessary and sufficient condition on $f(a)$ under which (U, ρ_f) is a representation of A.
We denote the set of maps f satisfying this condition by $Z^1(W, V)$.
This set is a vector space, and its elements are called (1-)cocycles. When $f \in Z^1(W, V)$, observe that (U, ρ_f) has a subrepresentation isomorphic to (V, ρ_V) and a quotient isomorphic to (W, ρ_W).
3. Let $F : W \to V$ be a linear map.
Define the coboundary of F to be the function $dF : A \to \mathrm{Hom}_{\mathbb{K}}(W, V)$ with the formula
$$dF(a) = \rho_V(a) \circ F - F \circ \rho_W(a) \quad \text{for } a \in A.$$
Show that $dF \in Z^1(W, V)$.
Check that $dF = 0$ if and only if F is a morphism of representations $(W, \rho_W) \to (V, \rho_V)$.
Let $B^1(W, V)$ be the subspace of coboundaries in $Z^1(W, V)$ and define $\mathsf{Ext}^1(W, V) = Z^1(W, V)/B^1(W, V)$.
4. Fix $f, g \in Z^1(W, V)$ and consider the A-representations (U, ρ_f) and (U, ρ_g).
Show that if $f - g \in B^1(W, V)$ then (U, ρ_f) and (U, ρ_g) are isomorphic.
5. Continue the notation of the previous exercise.
Further assume that V and W are finite dimensional and irreducible.
Show that (U, ρ_f) and (U, ρ_g) are isomorphic if and only if f and g represent elements of $\mathsf{Ext}^1(W, V)$ that are scalar multiples of each other.
6. Assume $\mathbb{K} = \mathbb{C}$ and $A = \mathbb{C}[x_1, x_2, \ldots, x_n]$.
Let $a = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$ and $b = (b_1, b_2, \ldots, b_n) \in \mathbb{C}^n$.
Suppose V_a and V_b are 1-dimensional A-representations in which x_i acts as a_i and b_i, respectively.
Find $\mathsf{Ext}^1(V_a, V_b)$ and classify the 2-dimensional representations of A.

Now we have some exercises about group representations.

7. Let $p > 0$ be a prime integer and suppose G is a finite group of size p^n for some integer $n \geq 1$. Show that the every irreducible representation of G over a field $\mathbb{K}$ of characteristic p is *trivial* (meaning that all elements of G act as the identity transformation).

8. Show that if $\mathbb{K}$ is an algebraically closed field of positive characteristic $p > 0$ and G is a finite group whose size is divisible by p, then the number of isomorphism classes of irreducible representations of G over $\mathbb{K}$ is strictly less than the number of conjugacy classes in G.

9. Show that a finite group is abelian if and only if all of its irreducible representations over $\mathbb{C}$ are 1-dimensional.

10. Let $n > 0$ be an integer and set $\theta = 2\pi/n$. Define $I_2(n)$ to be the subgroup of $\mathrm{GL}_2(\mathbb{C})$ generated by

$$r = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{and} \quad s = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Compute the size of $I_2(n)$ and describe its conjugacy classes.

Then describe all irreducible complex representations of $I_2(n)$ (up to isomorphism).

You should consider the cases of odd and even n separately.