

**Instructions:** Complete the following exercises.

Solutions must be hand-written and submitted in-person.

You will be graded on clarity and simplicity as well as correctness.

You may use any resources and work with other students, but you must write up your own solutions.

Due on **Tuesday, March 17**.

1. Let  $G$  be a finite group.

Suppose  $\chi_1, \chi_2, \dots, \chi_r$  are the distinct irreducible character of  $G$  over  $\mathbb{K} = \mathbb{C}$ .

For each  $i \in \{1, 2, \dots, r\}$  let

$$e_i = \frac{\chi_i(1)}{|G|} \sum_{g \in G} \chi_i(g) g^{-1} \in \mathbb{C}[G].$$

Prove that these elements are central idempotents that sum to 1 in  $\mathbb{C}[G]$ . In other words, show that  $1 = e_1 + e_2 + \dots + e_r$  and  $e_i^2 = e_i$  and  $e_i e_j = 0$  for  $i \neq j$  and  $e_i g = g e_i$  for all  $g \in G$ .

2. Let  $V$  be a finite dimensional complex vector space, and let  $\text{GL}(V)$  be the group of invertible linear transformations  $V \rightarrow V$ . Recall the definitions of the symmetric and exterior algebras

$$S^n V = \bigoplus_{n \geq 0} S^n V \quad \text{and} \quad \wedge^n V = \bigoplus_{n \geq 0} \wedge^n V.$$

Here  $S^n V$  and  $\wedge^n V$  are the images of  $T^n V$  under the relevant quotient map.

Both  $S^n V$  and  $\wedge^n V$  are representations of  $\text{GL}(V)$  in a natural way.

Show that  $S^n V$  is irreducible for all  $n \geq 0$  while  $\wedge^n V$  is irreducible if and only if  $0 \leq n \leq \dim V$ .

3. Let  $G$  be a finite group with a complex representation  $V$  of finite dimension. Assume this representation is *faithful*, meaning that the corresponding map  $G \rightarrow \text{GL}(V)$  is injective. Show that each irreducible representation of  $G$  occurs a subrepresentation of  $V^{\otimes n}$  for some  $n > 0$ .
4. Suppose  $K \subset H \subset G$  are groups and  $V$  is a representation of  $K$ . Show that  $\text{Ind}_H^G(\text{Ind}_K^H(V))$  is isomorphic to  $\text{Ind}_K^G(V)$  as  $G$ -representations.
5. Suppose  $H \subset G$  are finite groups and  $\chi : H \rightarrow \mathbb{C} \setminus \{0\} = \text{GL}_1(\mathbb{C})$  is a group homomorphism. Define

$$e_\chi = \frac{1}{|H|} \sum_{h \in H} \chi(h) h^{-1} \in \mathbb{C}[H].$$

Check that  $e_\chi e_\chi = e_\chi$  and show that  $\text{Ind}_H^G(V) \cong \mathbb{C}[G] e_\chi$  as  $G$ -representations for  $V = (\mathbb{C}, \chi)$ .

6. Let  $H \subset G$  be finite groups. Let  $V$  be a complex  $H$ -representation. The dual space  $V^*$  is then also an  $H$ -representation. Show that  $\text{Ind}_H^G(V^*) \cong (\text{Ind}_H^G(V))^*$  as  $G$ -representations.