Instructions: Complete the following exercises.

Solutions must be hand-written and submitted in-person.

You will be graded on clarity and simplicity as well as correctness.

You may use any resources and work with other students, but you must write up your own solutions.

Due on **Thursday, April 16**.

Some definitions:

- If R is a commutative ring and $r \in R$ then let (r) be the intersection of all ideals of R containing r.
- An *integral domain* is a commutative ring R such that if $r_1, r_2 \in R$ are both nonzero then $r_1 r_2 \neq 0$.
- A nonzero element r in an integral domain R is *irreducible* if whenever $r = r_1 r_2$ for some $r_i \in R$, exactly one of r_1 or r_2 is a *unit* (an element with a multiplicative inverse).

  A unit is not considered to be irreducible.
- A *PID (principal ideal domain)* is an integral domain where all ideals are generated by one element.
- A *maximal ideal* is a proper ideal that is not contained in any larger proper ideal.
- A *prime ideal* is a proper ideal that contains $r_1 r_2$ only when it contains at least one of r_1 or r_2.

1. Let R be a commutative ring with an ideal I.
   (a) Prove that R/I is an integral domain if and only if I is prime.
   (b) Prove that R/I is a field if and only if I is maximal.
   (c) Prove that if R is an integral domain then so is $R[x]$.
2. Let K be a field. Suppose $f, g \in K[x]$ and $g \neq 0$.

   Prove that there are two polynomials $q, r \in K[x]$ with $f = gq + r$ and $\deg(r) < \deg(g)$.

   Also prove that q and r are uniquely determined by these properties.

   Finally, use these facts to deduce that $f(a) = 0$ for $a \in K$ if and only if $x - a$ divides $f(x)$.
3. Suppose K is a field. Suppose $I \subseteq K[x]$ is a nonzero ideal.

   Use the preceding exercise to prove that there is a unique monic polynomial $P(x) \in I$ with
   $$\deg(P(x)) = \min\{\deg(f) : f \in I\}$$
   such that $(P(x)) = I$. Conclude that $K[x]$ is a PID.
4. A *UFD (unique factorization domain)* is an integral domain R such that if $0 \neq r \in R$ then
   (a) there is sequence of irreducible elements $r_1, \ldots, r_k \in R$ such that $(r) = (r_1 \cdots r_k)$, and
   (b) if $s_1, \ldots, s_l \in R$ is another such sequence then $k = l$ and there is a permutation $\sigma \in S_k$ and a sequence of unit elements $u_1, \ldots, u_k \in R$ such that $u_i r_i = s_{\sigma(i)}$ for all $i = 1, 2, \ldots, k$.

   It is known that every PID is a UFD.

   Look up a proof of this fact and sketch the steps. (You don't have to give complete details.)

   Then find an example of a UFD that is not a PID. (Prove your answer to this part.)
5. Suppose that R is a UFD. Prove that $0 \neq f \in R$ is irreducible if and only if (f) is a prime ideal.
6. Suppose that R is a PID. Prove that a nonzero ideal is prime if and only if it is maximal.