Instructions: Complete the following exercises.

Solutions must be hand-written and submitted in-person.

You will be graded on clarity and simplicity as well as correctness.

You may use any resources and work with other students, but you must write up your own solutions.

Due on **Thursday, May 7**.

1. Prove that any finite subgroup of the multiplicative group of a field is cyclic.
2. Suppose $M \mid L$ and $L \mid K$ are field extensions.

   Prove that the following properties are equivalent:

   (a) $M \mid K$ is algebraic.

   (b) $M \mid L$ and $L \mid K$ are algebraic.
3. Let K be a field with $p = \mathrm{ch}(K)$. Suppose $L \mid K$ is a field extension of finite degree n.

   (a) Prove that an irreducible $f(x) \in K[x]$ is not separable if and only if $p > 0$ and $f(x) \in K[x^p]$.

   (b) Use part (a) to show that if p does not divide n then $L \mid K$ is separable.
4. Let K be a field of characteristic $p > 0$ suppose $a \in K$ is such that $f(x) = x^p - x - a$ is irreducible.

   Suppose β is a root of $f(x) \in K[x]$ in some extension of K.

   Prove that $K(\beta) \mid K$ is a Galois extension and compute its Galois group.
5. Let $K = \mathbb{F}_2[x]/(x^3 + x + 1)$ and $L = \mathbb{F}_2[x]/(x^3 + x^2 + 1)$.

   Prove that K and L are fields and then explicitly describe all isomorphisms $K \cong L$.
6. Suppose p is prime and ω_n is a primitive nth root of unity in some extension of $\mathbb{F}_p$.

   Compute $|\mathbb{F}_p(\omega_n)|$.