1 Review: algebras, representations, and Schur's lemma

Let $\mathbb{K}$ be a field, assumed to be algebraically closed unless noted otherwise.

Definition. An *(associative, unital) algebra* is a nonzero $\mathbb{K}$-vector space A with a bilinear *product* map $A \times A \to A$ and a *unit* element $1 \in A$ that satisfy $a(bc) = (ab)c$ and $1a = a1 = a$ for all $a, b, c \in A$.

A *morphism* $f : A \to B$ of algebras is a $\mathbb{K}$-linear map with $f(1) = 1$ and $f(ab) = f(a)f(b)$ for all $a, b \in A$.

A morphism that is a bijection is called an *isomorphism*.

Example. If V is a nonzero $\mathbb{K}$-vector space, then the vector space $\mathsf{End}(V)$ of all linear maps $V \to V$ is an algebra, where the product is composition $\rho_1\rho_2 = \rho_1 \circ \rho_2$ and the unit is the identity map $\mathrm{id}_V : V \to V$.

Definition. A *representation* of an algebra A is a pair (ρ, V) where

- V is a $\mathbb{K}$-vector space, and
- ρ is a linear map $A \to \mathsf{End}(V)$ satisfying $\rho(1) = \mathrm{id}_V$ and $\rho(ab) = \rho(a)\rho(b)$ for all $a, b \in A$.

A *morphism* $\phi : (\rho_1, V_1) \to (\rho_2, V_2)$ of representations of A is a linear map $\phi : V_1 \to V_2$ with

$$\phi(\rho_1(a)(v)) = \rho_2(a)(\phi(v)) \qquad \text{for all } a \in A \text{ and } v \in V_1.$$

This is sometimes called an *intertwining operator*. A morphism that is a bijection is an *isomorphism*.

Definition. A *subrepresentation* of (ρ, V) is a subspace $W \subseteq V$ with $\rho(a)(W) \subseteq W$ for all $a \in A$.

If W is a subrepresentation, then we view the pair (ρ, W) as another representation of A.

We say that (ρ, V) is *irreducible* if $V \neq 0$ and there are no other subrepresentations except V and 0.

Proposition (*Schur's Lemma*)**.** Let $\phi : (\rho_1, V_1) \to (\rho_2, V_2)$ be a morphism of representations of A.

(a) If both representations are irreducible then ϕ is an isomorphism (even if $\mathbb{K}$ is not alg. closed).

(b) If $(\rho, V) = (\rho_1, V_1) = (\rho_2, V_2)$ is irreducible with $\dim(V) < \infty$, then $\phi = \lambda \cdot \mathrm{id}_V$ for some $\lambda \in \mathbb{K}$.

(c) If A is *commutative* (so $ab = ba$ for all $a, b \in A$) then every irreducible repn (ρ, V) has $\dim(V) = 1$.

Example. Properties (b) and (c) can both fail if $\mathbb{K}$ is not algebraically closed.

We can see in this in examples when $\mathbb{K} = \mathbb{R}$ is the (not algebraically closed) field of real numbers:

(i) Let $A = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}$. As an $\mathbb{R}$-algebra we have $A \cong \mathbb{C}$.

Let $\rho : A \to \mathsf{End}(A)$ be the (left) regular representation of A, so that $\rho(y)(z) = yz$.

Every $0 \neq z \in A$ is invertible, so (ρ, V) is irreducible but $\dim(V) = 2$.

Thus, even though A is commutative, not all of its irreducible representations are 1-dimensional.

(ii) Now define $\phi : A \to A$ by $\phi\left(\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \right) = \begin{bmatrix} -b & -a \\ a & -b \end{bmatrix}$, which is multiplication by $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

This is a morphism $\phi : (\rho, V) \to (\rho, V)$ since A is commutative, but it is not a scalar map.

2 Indecomposable representations

Let A be an algebra over $\mathbb{K}$. In this section $\mathbb{K}$ does not need to be algebraically closed.

Suppose (ρ_1, V_1) and (ρ_2, V_2) are representations of A.

Then we can form the *direct sum representation*

$$(\rho_1, V_1) \oplus (\rho_2, V_2) \stackrel{\text{def}}{=} (\rho_1 \oplus \rho_2, V_1 \oplus V_2)$$

where $V_1 \oplus V_2 = \{v_1 + v_2 : v_1 \in V_1 \text{ and } v_2 \in V_2\}$ is the usual vector space direct sum and

$$\rho_1 \oplus \rho_2 : A \to \mathsf{End}(V_1 \oplus V_2)$$

is defined by the formula

$$(\rho_1 \oplus \rho_2)(a)(v_1 + v_2) = \rho_1(a)(v_1) + \rho_2(a)(v_2) \qquad \text{for } a \in A,\ v_1 \in V_1 \text{ and } v_2 \in V_2.$$

Notice that $(\rho_1, V_1) \oplus (\rho_2, V_2) \cong (\rho_2, V_2) \oplus (\rho_1, V_1)$ via the vector space isomorphism $V_1 \oplus V_2 \cong V_2 \oplus V_1$.

Definition. A representation (ρ, V) of A is *indecomposable* if it is not isomorphic to $(\rho_1, V_1) \oplus (\rho_2, V_2)$ for any nonzero representations (ρ_i, V_i) of A.

This occurs if and only if (ρ, V) does not have nonzero subrepresentations $W_1, W_2 \subseteq V$ with $V = W_1 \oplus W_2$.

Notation. If $W_1 \subseteq V$ and $W_2 \subseteq V$ are subspaces then writing

(a) $V = W_1 \oplus W_2$

is just an abbreviation for the property

(b) it holds that $V = W_1 + W_2$ and $0 = W_1 \cap W_2$.

Formally, the direct sum $W_1 \oplus W_2$ is some new vector space equipped with inclusions

$$W_1 \to W_1 \oplus W_2 \leftarrow W_2$$

satisfying a universal property. When (b) holds, V has this property so can be identified with $W_1 \oplus W_2$.

Note that irreducible $\implies$ indecomposable, but not vice versa.

Example. Consider the (commutative) polynomial algebra $\mathbb{K}[x]$.

What are the irreducible representations of $\mathbb{K}[x]$?

Choose a linear map $L : V \to V$ where V is a vector space.

Define $\rho_L : \mathbb{K}[x] \to \mathsf{End}(V)$ by formula such that $\rho_L(f(x)) = f(L)$ so that

$$\rho_L(a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0) = a_n L^n + \ldots + a_2 L^2 + a_1 L + a_0 I.$$

Then (ρ_L, V) is a representation of $\mathbb{K}[x]$.

Every representation of $\mathbb{K}[x]$ must arise via this construction.

This holds as every algebra morphism $\mathbb{K}[x] \to B$ is uniquely determined by the image of the variable x.

It is possible that different choices of L might give isomorphic representations (ρ_L, V), however.

Earlier results tell us that (ρ_L, V) is irreducible if and only if $\dim V = 1$.

What are the indecomposable representations of $\mathbb{K}[x]$?

Choose $\lambda \in \mathbb{K}$ and an integer $n \geq 1$. Define $J_{\lambda,n} : \mathbb{K}^n \to \mathbb{K}^n$ to be the linear map

$$J_{\lambda,n} = \begin{bmatrix} \lambda & 1 & & & 0 \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & \lambda \end{bmatrix}.$$

Then $(\rho_{J_{\lambda,n}}, \mathbb{K}^n)$ is indecomposable, though not irreducible if $n > 1$.

Moreover, every indecomposable representation of $\mathbb{K}[x]$ is isomorphic to one of these representations.

This follows by the uniqueness of Jordan canonical form.

Finally, one can show that $(\rho_{J_{\lambda,n}}, \mathbb{K}^n) \cong (\rho_{J_{\lambda',n'}}, \mathbb{K}^{n'})$ if and only if $n = n'$ and $\lambda = \lambda'$.

(These statements are not self-evident and require a fair amount of linear algebra to derive.)

3 Group Representations

Suppose G is a group. Recall that the group algebra is $\mathbb{K}[G] = \mathbb{K}\text{-span}\{a_g : g \in G\}$ where $a_g a_h = a_{gh}$.

Given a vector space V, let $\mathrm{GL}(V) \subset \mathsf{End}(V)$ be the group of invertible linear maps $V \to V$.

Definition. A *group representation* of G is a pair (ρ, V) where V is a vector space and

$$\rho : G \to \mathrm{GL}(V)$$

is a group homomorphism.

We claim that group representations are the same as representations of the corresponding group algebra.

In one direction, any group representation (ρ, V) for G becomes a representation of $\mathbb{K}[G]$ on setting

$$\rho(a_g) = \rho(g) \qquad \text{for } g \in G$$

and extending by linearity.

In the other direction, if (ρ, V) is a representation of $\mathbb{K}[G]$ then $\rho(a_g) \in \mathrm{GL}(V)$ for all $g \in G$ as

$$\rho(a_g)\rho(a_{g^{-1}}) = \rho(a_g a_{g^{-1}}) = \rho(a_1) = \mathrm{id}_V .$$

Hence $g \mapsto \rho(a_g) \in \mathrm{GL}(V)$ is a group homomorphism $G \to \mathrm{GL}(V)$.

4 Ideals in algebras

Let A be an algebra. If $a \in A$ and $S \subseteq A$ then let $aS = \{ab : b \in S\}$ and $Sa = \{ba : b \in S\}$.

Definition. A *left ideal* in A is a subspace $I \subseteq A$ with $aI \subseteq I$ for all $a \in A$.

A *right ideal* in A is a subspace $I \subseteq A$ with $Ia \subseteq I$ for all $a \in A$.

A *two-sided ideal* in A is a subspace that is both a left and right ideal.

All three notions coincide if A is commutative.

The left ideals of A are precisely the subrepresentations of the regular representation of A.

Similarly, the right ideals of A are the subrepresentations of the regular representation of A^{op}.

The subspaces 0 and A are always two-sided ideals. If these are the only two-sided ideals then A is *simple*.

Example. The algebra $\mathsf{Mat}_{n\times n}(\mathbb{K})$ is simple.

To check this, suppose $I \subseteq \mathsf{Mat}_{n\times n}(\mathbb{K})$ is a nonzero two-sided ideal.

We need to show that every $n \times n$ matrix is in I.

If there is some elementary matrix $E_{jk} \in I$, then every other elementary matrix is obtained as

$$E_{il} = E_{ij}E_{jk}E_{kl} \in I$$

so any linear combination of elementary matrices is in I, which means that every $n \times n$ matrix is in I.

Thus it is enough to show that I contains some elementary matrix.

We get this as I is nonzero, so there is some $0 \neq M \in I$ with some nonzero entry $M_{jk} \neq 0$, and then

$$E_{jk} = \tfrac{1}{M_{jk}} E_{jj} M E_{kk} \in I.$$

Example. If $\phi : A \to B$ is an algebra morphism then its *kernel* is the set

$$\ker(\phi) = \{a \in A : \phi(a) = 0\}$$

The kernel is always a subspace, and if $a \in \ker(\phi)$ and $x, y \in A$ then

$$\phi(xa) = \phi(x)\phi(a) = 0 \qquad \text{and} \qquad \phi(ay) = \phi(a)\phi(y) = 0$$

so $xa \in \ker(\phi)$ and $ay \in \ker(\phi)$. Thus shows that $\ker(\phi)$ is a two-sided ideal.

Example. If $S \subseteq A$, then we define $\langle S\rangle$ to be the intersection of all two-sided ideals in A containing S.

We call this the *two-sided ideal generated by* S. One can show that each element of $\langle S\rangle$ has the form

$$a_1s_1b_1 + a_2s_2b_2 + \cdots + a_ns_nb_n$$

for some $n \geq 0$ and some choice of $a_i, b_i \in A$ and $s_i \in S$.

Definition. A *maximal* left/right/two-sided ideal $I \subsetneq A$ is an ideal properly contained in exactly one other left/right/two-sided ideal (namely A itself).

One can use *Zorn's lemma* to show that every ideal is contained in a maximal ideal.

(Zorn's lemma is only needed if A is infinite-dimensional.)

Definition. Assume I is two-sided ideal in an algebra A with $I \neq A$. Then the quotient vector space

$$A/I = \{a + I : a \in A\}$$

where $a + I \overset{\text{def}}{=} \{a + i : i \in I\}$ is an algebra with unit $1 + I$ for the multiplication defined by

$$(a + I)(b + I) = ab + I \quad \text{for } a, b \in A.$$

There is something to check to make sure that the above multiplication is well-defined.

This is a standard exercise. The linear map $\pi : A \to A/I$ with $\pi(a) = a + I$ is an algebra morphism.

Definition. Suppose (ρ_V, V) is a representation of A and $W \subseteq V$ is a subrepresentation.

Define $\rho_{V/W} : A \to \mathsf{End}(V/W)$ by the formula

$$\rho_{V/W}(a)(x + W) = \rho_V(a)(x) + W \qquad \text{for } a \in A \text{ and } x \in V.$$

Then $(\rho_{V/W}, V/W)$ is a representation of A, called the *quotient representation*.

If $I \subseteq A$ is a left ideal, then A/I is a representation of A via this construction.

Equivalently, A/I is a left A-module for the action $a \cdot (b + I) \overset{\text{def}}{=} ab + I$ for $a, b \in A$.

5 Generators and relations

Recall that $\mathbb{K}\langle X_1, X_2, \dots, X_n\rangle$ is the *free algebra* of polynomials in noncommuting variables.

If $f_1, f_2, \dots, f_m \in \mathbb{K}\langle X_1, X_2, \dots, X_n\rangle$ then we can consider the quotient algebra

$$\mathbb{K}\langle X_1, X_2, \dots, X_n\rangle / \langle f_1, f_2, \dots, f_m\rangle$$

where $\langle f_1, f_2, \dots, f_m\rangle$ is the two-sided ideal generated by $\{f_1, f_2, \dots, f_m\}$. We often write this as

$$\mathbb{K}\langle X_1, X_2, \dots X_n \mid f_1 = f_2 = \dots f_m = 0\rangle.$$

The elements of this quotient are polynomials where we can replace expressions equal to f_i by zero.

Remark. Technically, if $I = \langle f_1, f_2, \dots, f_m\rangle$ then elements of $\mathbb{K}\langle X_1, X_2, \dots, X_n\rangle/I$ are cosets $f + I$.

Usually we write things by dropping the "$+I$" part.

When we do this it will be clear from context whether f belongs to $\mathbb{K}\langle X_1, X_2, \dots, X_n\rangle$ or the quotient.

Example. The *Weyl algebra* is the quotient algebra

$$\mathbb{K}\langle x, y \mid yx - xy - 1 = 0\rangle = \mathbb{K}\langle x, y \mid yx - xy = 1\rangle.$$

In the Weyl algebra, we have $yx = xy + 1$ and $xyx = x(xy + 1) = x^2y + x = (yx - 1)x = yx^2 - x$.

Example. The *q-Weyl algebra* for a fixed nonzero element $q \in \mathbb{K}$ is

$$\mathbb{K}\left\langle x, x^{-1}, y, y^{-1} \mid yx = qxy \text{ and } xx^{-1} = x^{-1}x = yy^{-1} = y^{-1}y = 1\right\rangle.$$

The second set of relations ensures that x, x^{-1} and y, y^{-1} are inverses of each other.