1 Review from last time

Let A be an algebra over a field $\mathbb{K}$, which is algebraically closed unless otherwise noted.

Given two representations (ρ_1, V_1) and (ρ_2, V_2) of A, we can form the *direct sum representation*

$$(\rho_1, V_1) \oplus (\rho_2, V_2) \overset{\text{def}}{=} (\rho_1 \oplus \rho_2, V_1 \oplus V_2)$$

where $(\rho_1 \oplus \rho_2)(a)(v_1 + v_2) = \rho_1(a)(v_1) + \rho_2(a)(v_2)$ for $v_1 \in V_1$, $v_2 \in V_2$, $a \in A$.

A representation of A is *indecomposable* if it is not a direct sum of two nonzero representations.

If $(\rho, V) \cong (\rho_1, V_1) \oplus (\rho_2, V_2)$ then each V_i is a subrepresentation.

Thus, for representations of A, the property NOT indecomposable $\implies$ NOT irreducible:

Taking contrapositives shows that $\boxed{\text{irreducible} \implies \text{indecomposable}}$.

Given any subspace $I \subseteq A$, let $A/I = \{a + I : a \in A\}$ be the vector space quotient.

A subspace I is a *two-sided ideal* in A if $axb \in I$ for all $a, b \in A$ and $x \in I$.

If $I \neq A$ is a two-sided ideal then A/I is an algebra, with product $(a + I)(b + I) = ab + I$ for $a, b \in A$.

Quotient algebras are useful since they let us define algebras by generators and relations.

Example. The *Weyl algebra* is $\mathbb{K}\langle x, y \mid yx - xy = 1\rangle$.

This notation means the quotient of free algebra $\mathbb{K}\langle x, y\rangle$ by the two-sided ideal

$$\langle yx - xy - 1\rangle \overset{\text{def}}{=} (\text{the intersection of all two-sided ideals containing } yx - xy - 1).$$

To save space, we write $f(x, y)$ instead of $f(x, y) + \langle yx - xy - 1\rangle$ to denote elements of this algebra.

It often hard to say concretely what an ideal like $\langle yx-xy-1\rangle$ is explicitly, and to classify precisely which expressions in $\langle x, y\rangle$ become zero in the quotient.

Taking quotients means we can make substitutions like $yx = xy + 1$ in polynomial expressions.

The relations in an algebra defined by generators and relations provide an algorithm for transforming a given expression to others that are equal in the algebra. In principle, an exhaustive search using this algorithm can tell you if two expressions are equal, but this search might not terminate.

2 More on the Weyl algebra

The elements of the Weyl algebra $\mathbb{K}\langle x, y \mid yx = xy + 1\rangle$ technically are cosets of the ideal $\langle yx - xy - 1\rangle$.

However, for simplicity we just write these elements as ordinary polynomials in x and y.

Proposition. A basis for the Weyl algebra is $\{x^i y^j : i, j \geq 0\}$.

Proof. The given set spans the algebra since we can always transform

$$x^{i_1}y^{j_1}x^{i_2}y^{j_2}\dots x^{i_k}y^{j_k} = x^{i_1+i_2+\dots+i_k}y^{j_1+j_2+\dots+j_k} + (\text{lower degree terms})$$

via repeated substitutions $yx = xy + 1$.

To show linear independence, assume $\mathrm{ch}(\mathbb{K}) = 0$.

(The argument when $\mathrm{ch}(\mathbb{K}) > 0$ is similar but not as elegant; see the textbook for details.)

Consider the polynomial ring $\mathbb{K}[z]$. For $f \in \mathbb{K}[z]$, define $x \cdot f = zf$ and $y \cdot f = \frac{d}{dz} f$.

There is a unique left module structure (for the Weyl algebra) on $\mathbb{K}[z]$ with these formulas, because

$$y \cdot (x \cdot f) = y \cdot (zf) = \tfrac{d}{dz}(zf) = f + z \tfrac{d}{dz} f = f + x \cdot (y \cdot f),$$

which means that $(yx - xy - 1) \cdot f = 0$.

Now suppose $c_{ij} \in \mathbb{K}$ are coefficients such that $\sum_{i,j \geq 0} c_{ij} x^i y^j = 0$ in the Weyl algebra.

Let $L = \sum_{i,j\geq 0} c_{ij} z^i \left(\frac{d}{dz}\right)^j$ be a differential operator on $\mathbb{K}[z]$. Then

$$L(f) = \left(\textstyle\sum_{i,j \geq 0} c_{ij} x^i y^j\right) \cdot f = 0 \quad \text{for all } f \in \mathbb{K}[z].$$

But we can write $L = \sum_{j=0}^{r} Q_j(z) \left(\frac{d}{dz}\right)^j$ for some polynomials $Q_j(z) \in \mathbb{K}[z]$.

Now observe that

$$\begin{aligned} &L(1) = Q_0(z) = 0 \\ &L(z) = Q_0(z)z + Q_1(z) = Q_1(z) = 0 \\ &L(z^2) = Q_0(z)z^2 + Q_1(z)z + Q_2(z) = Q_2(z) = 0 \\ &\vdots \end{aligned}$$

Thus we have $Q_0 = Q_1 = \ldots = Q_r = 0 \implies c_{ij} = 0$ for every i, j.

This proves that the elements $x^i y^j$ for $i, j \geq 0$ are linearly independent. □

Example. Let $q \in \mathbb{K}$ be a nonzero element. Then the *q-Weyl algebra* is

$$A = \mathbb{K}\left\langle x, x^{-1}, y, y^{-1} \mid yx = qxy,\ xx^{-1} = x^{-1}x = 1,\ yy^{-1}y^{-1}y = 1\right\rangle.$$

We require q to be nonzero, since if $q = 0$ then we would have $x = y = 0$ and $A = 0$:

$$yx = 0 \implies y^{-1}yx = x = 0 \text{ and } yxx^{-1} = y = 0.$$

Proposition. If $q \neq 0$ then a basis for the q-Weyl algebra is $\{x^i y^j : i, j \in \mathbb{Z}\}$

Proof. The argument to show that given elements span the algebra is similar to the Weyl algebra case.

For linear independence, see the textbook. □

3 Tensor products of vector spaces

Let V and W be two $\mathbb{K}$-vector spaces. Their direct product is simply the set of pairs

$$V \times W = \{(v, w) : v \in w,\ b \in W\}.$$

This object is just a set, not a vector space.

Define the *free product* $V * W$ to be the $\mathbb{K}$-vector space with $V \times W$ as a basis.

Each element of $V * W$ is a finite linear combination of pairs $(v, w) \in V \times W$.

One way to define the *tensor product* of V and W is as the quotient vector space

$$V \otimes W \stackrel{\text{def}}{=} (V * W)/\mathcal{I}_{V,W}$$

where $\mathcal{I}_{V,W}$ is the subspace spanned by all elements of the form

- $(v_1 + v_2, w) - (v_1, w) - (v_2, w)$,
- $(v, w_1 + w_2) - (v, w_1) - (v, w_2)$,
- $(av, w) - a(v, w)$, or
- $(v, aw) - a(v, w)$,

for any $a \in \mathbb{K}$, $v_1, v_2, v \in V$, and $w_1, w_2, w \in W$.

This construction comes with a quotient map $V * W \to V \otimes W$.

If $x \in V$ and $y \in W$, then let $x \otimes y \in V \otimes W$ be the image of $(x, y) \in V \times W \subset V * W$ under this map.

If we view elements of a vector space quotient as cosets of a subspace then

$$x \otimes y \stackrel{\text{def}}{=} (x, y) + \mathcal{I}_{V,W}$$

We refer to $x \otimes y$ as a *pure tensor*.

Not all elements of $V \otimes W$ are pure tensors, but they are a spanning set for $V \otimes W$.

We can manipulate pure tensors without changing their value in $V \otimes W$ using the following identities:

$$\begin{aligned} (v_1 + v_2) \otimes w &= v_1 \otimes w + v_2 \otimes w \\ v \otimes (w_1 + w_2) &= v \otimes w_1 + v \otimes w_2 \end{aligned} \qquad \text{and} \qquad (cv) \otimes w = c(v \otimes w) = v \otimes (cw)$$

for $v_1, v_2, v \in V$ and $w_1, w_2, w \in W$ and $c \in \mathbb{K}$.

These equations hold because the differences between the two sides belong to the subspace $\mathcal{I}_{V,W}$.

We can have $x \otimes y = x' \otimes y'$ when $x \neq x'$ and $y \neq y'$.

A simple example is when $x' = -x \in V$ and $y' = -y \in W$.

Exercise (Should do once)**.** Suppose $\{v_i : i \in I\}$ is a basis of V and $\{w_j : j \in J\}$ is a basis of W.

Then the set $\{v_i \otimes w_j : (i, j) \in I \times J\}$ is a basis of $V \otimes W$.

Remark. If U, V, and W are $\mathbb{K}$-vector spaces, then there is a unique isomorphism

$$(U \otimes V) \otimes W \xrightarrow{\sim} U \otimes (V \otimes W)$$

that sends $u \otimes (v \otimes w) \mapsto (u \otimes v) \otimes w$ for each $u \in U$, $v \in V$, and $w \in W$.

It follows from this exercise that there is a canonical isomorphism between any way of forming the tensor product between a finite sequence of vector spaces. For example:

$$V_1 \otimes ((V_2 \otimes V_3) \otimes V_4) \cong V_1 \otimes (V_2 \otimes (V_3 \otimes V_4)) \cong (V_1 \otimes V_2) \otimes (V_3 \otimes V_4) \cong ((V_1 \otimes V_2) \otimes V_3) \otimes V_4.$$

In view of this, we will ignore the issue of parenthesization and just define

$$V^{\otimes 0} = \mathbb{K} \quad \text{and} \quad V^{\otimes n} = V \otimes V \otimes \cdots \otimes V \text{ (n factors)}.$$

4 Tensor products of linear maps

If V and W are vector spaces over $\mathbb{K}$ then let $\mathrm{Hom}(V, W)$ be the vector space of linear maps $V \to W$.
If $f \in \mathrm{Hom}(V, V')$ and $g \in \mathrm{Hom}(W, W')$ are two linear maps then their *tensor product*

$$f \otimes g : V \otimes W \to V' \otimes W'$$

is the unique linear map that acts on pure tensors as

$$v \otimes w \mapsto f(v) \otimes g(w) \quad \text{for all } v \in V \text{ and } w \in W.$$

There are some things to check to make sure that this is well-defined.
Since $V \times W$ is a basis for $V * W$, there is a unique linear map $f * g : V * W \to V' \otimes W'$ that sends

$$(v, w) \mapsto f(v) \otimes g(w) \quad \text{for all } v \in V \text{ and } w \in W.$$

We want know that the map $f * g$ descends to a well-defined map of quotient spaces $V \otimes W \to V' \otimes W'$.
Thus will give exactly our desired map $f \otimes g$.

We need to verify that $(f * g)(\mathcal{I}_{V,W}) = 0 \subseteq V' \otimes W'$.
To check this, it is enough to show that $f * g$ sends each element in the spanning set for $\mathcal{I}_{V,W}$ to zero.
This is routine algebra. For instance, if $v_1, v_2 \in V$ and $w \in W$ then we have

$$\begin{aligned}(f * g)((v_1 + v_2, w) - (v_1, w) - (v_2, w)) &= (f * g)((v_1 + v_2, w)) - (f * g)((v_1, w)) - (f * g)((v_2, w)) \\ &= f(v_1 + v_2) \otimes g(w) - f(v_1) \otimes g(w) - f(v_2) \otimes g(w) \\ &= f(v_1) \otimes g(w) + f(v_2) \otimes g(w) - f(v_1) \otimes g(w) - f(v_2) \otimes g(w) \\ &= 0\end{aligned}$$

as needed. The calculations showing that $f * g$ kills off the other elements spanning $\mathcal{I}_{V,W}$ are similar.

5 Tensor algebras

Definition. The *tensor algebra* of a vector space V is the infinite direct sum $\mathcal{T}V = \bigoplus_{n \geq 0} V^{\otimes n}$.
The elements of an infinite direct sum are finite sums of elements from the summands.

We view $\mathcal{T}V$ as a $\mathbb{K}$-algebra by defining

$$ab = a \otimes b \quad \text{for } a \in V^{\otimes m} \text{ and } b \in V^{\otimes n},$$

and extending by bilinearity. Here we view $a \otimes b \in V^{\otimes (m+n)}$.
This product is associative, since the tensor product is associative.
The unit of the resulting *tensor algebra* $\mathcal{T}V$ is the field unit $1 = 1_{\mathbb{K}} \in \mathbb{K} = V^{\otimes 0}$.

Notice that $\mathcal{T}V$ is an algebra even when $V = 0$, since then $\mathcal{T}V = \mathcal{T}0 = \mathbb{K}$.

Remark. We may identify tensor algebras with free algebras.

Suppose V is finite-dimensional with basis $\{v_1, \dots, v_N\}$. Then there is a unique algebra isomorphism

$$\mathcal{T}V \xrightarrow{\sim} \mathbb{K}\langle X_1, \dots, X_N \rangle$$

that sends $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k} \mapsto X_{i_1} X_{i_2} \cdots X_{i_k}$.

A similar isomorphism exists when $\dim(V) = \infty$, if we allow infinitely-many variables in the free algebra.

6 Symmetric algebras

We mention two interesting quotients of the tensor algebra.

Definition. The *symmetric algebra* of V is the quotient $\mathcal{S}V = \mathcal{T}V / \langle v \otimes w - w \otimes v : v, w \in V \rangle$.

Recall that $\langle v \otimes w - w \otimes v : v, w \in V \rangle$ means the intersection of all two-sided ideals in $\mathcal{T}V$ containing

$$\{v \otimes w - w \otimes v : v, w \in V\}.$$

The symmetric algebra $\mathcal{S}V$ is always commutative. We have $\mathcal{T}V \cong \mathcal{S}V$ if and only if $\dim V \leq 1$.

Remark. We may identify symmetric algebras with polynomial algebras.

Suppose V is finite-dimensional with basis $\{v_1, \dots, v_N\}$. Then there is a unique algebra isomorphism

$$\mathcal{S}V \xrightarrow{\sim} \mathbb{K}[x_1, \dots, x_N]$$

that sends $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k} \mapsto x_{i_1} x_{i_2} \cdots x_{i_k}$.

A similar isomorphism exists when $\dim(V) = \infty$.

7 Exterior algebras

Definition. The *exterior algebra* of V is the quotient $\bigwedge V = \mathcal{T}V / \langle v \otimes v : \ v \in V \rangle$.

Define $x \wedge y$ to be the image $x \otimes y \in V \otimes V$ under the quotient map $\mathcal{T}V \to \bigwedge V$. Then

$$\begin{aligned} 0 &= (x+y) \wedge (x+y) \\ &= x \wedge x + x \wedge y + y \wedge x + y \wedge y \\ &= x \wedge y + y \wedge x \end{aligned}$$

so $x \wedge y = -y \wedge x$. This shows that the operation $\wedge$ is an *anti-commutative* product for $\bigwedge V$.

Remark. Choosing a basis for V determines an isomorphism from $\bigwedge V$ to a "polynomial algebra" in which the variables anti-commute in the sense that $x_i x_j = -x_j x_i$.

8 Tensor product of modules

Building on our definition of vector space tensor products, we can now define more general tensor products of modules over a (not necessarily commutative) algebra.

8.1 Right modules with left modules

Consider the following setup:

A, B, C are algebras over the same field $\mathbb{K}$,

V is a right B-module, and

W is a left B-module.

Then we define $V \otimes_B W$ to be the vector space quotient

$$V \otimes_B W \overset{\text{def}}{=} (V \otimes W)/\mathbb{K}\text{-span}\{vb \otimes w - v \otimes bw : v \in V,\ w \in W,\ b \in B\}.$$

In general, this object only has the structure of a $\mathbb{K}$-vector space.

Specifically, if B is non-commutative, then $V \otimes_B W$ is not naturally a left or right module for B.

We refer to $V \otimes_B W$ as the *tensor product of V and W over B*.

If $v \in V$ and $w \in W$ then we write

$$v \otimes_B w \in V \otimes_B W$$

for the image of $v \otimes w \in V \otimes W$ under the quotient map $V \otimes W \to V \otimes_B W$.

Notice that if $b \in B$ then $vb \otimes_B w = v \otimes_B bw$.

8.2 Bimodules

We now assume in addition that V is an *(A, B)-bimodule*, meaning that

V has **both** right B-module and left A-module structures, and

these structures are compatible in the sense that $(av)b = a(vb)$ for all $a \in A$, $b \in B$, $v \in V$.

Assume likewise that W is a *(B, C)-bimodule*, meaning that

W has **both** left B-module and right C-module structures, and

these structures are compatible in the sense that $(bw)c = b(wc)$ for all $b \in B$, $c \in C$, $w \in W$.

Then the vector space $V \otimes_B W$ has a (A, C)-bimodule structure defined by

$$a(v \otimes_B w) = (av) \otimes_B w \quad \text{and} \quad (v \otimes_B w)c = v \otimes_B (wc) \quad \text{for } a \in A,\ c \in C,\ v \in V,\ w \in W.$$

The case when $A = B = C$ is worth noting.

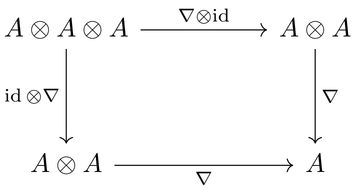
In this situation, V and W are both (B, B)-bimodules, and $V \otimes_B W$ is also a (B, B)-bimodule.

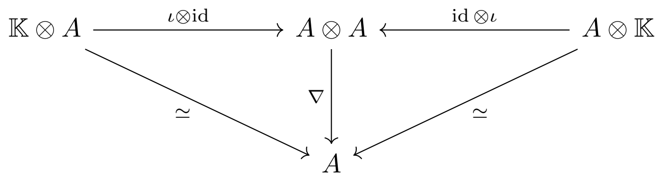
Remark. If the algebra B is commutative, then left and right B-modules are the same as (B, B)-bimodules, and so we can form the tensor product of two left B-modules or two right B-modules.

However, this is secretly just doing the (B, B)-bimodule tensor product described above.

9 Diagrammatic definition of an algebra

Now that we have a good handle on vector space tensor products, we can give an alternate definition of an *algebra*. This consists of a $\mathbb{K}$-vector space A with linear maps $\nabla : A \otimes A \to A$ and $\iota : \mathbb{K} \to A$ that make the following diagrams commute:





The diagonal arrows on the right are the linear maps $\mathbb{K} \otimes A \to A$ and $A \otimes \mathbb{K} \to A$ sending $1_{\mathbb{K}} \otimes a \mapsto a$ and $a \otimes 1_{\mathbb{K}} \mapsto a$ for all $a \in A$. These maps are vector space isomorphisms.

Under this formulation, the product in A is $ab \overset{\text{def}}{=} \nabla(a \otimes b)$ and the unit is $\iota(1_{\mathbb{K}}) \in A$.

One nice feature of this definition is that it naturally suggests the definition of a *coalgebra*: this is the object one gets by repeating the above definition but reversing the direction of all arrows.