1 Review: tensor products and tensor algebras

Let V and W be vector spaces defined over an arbitrary field $\mathbb{K}$.

- The *direct product* $V \times W$ is the set of pairs (v, w) with $v \in V$ and $w \in W$.
- The *free product* $V * W$ is the $\mathbb{K}$-vector space with $V \times W$ as a basis.
- The *tensor product* $V \otimes W$ is the quotient vector space

$$V \otimes W \stackrel{\text{def}}{=} (V * W)/\mathcal{I}_{V,W}$$

  where $\mathcal{I}_{V,W}$ is the subspace of $V * W$ spanned by the elements of the following forms:

  $(v_1 + v_2, w) - (v_1, w) - (v_2, w)$,

  $(v, w_1 + w_2) - (v, w_1) - (v, w_2)$,

  $(av, w) - a(v, w)$, and

  $(v, aw) - a(v, w)$,

  for any $a \in \mathbb{K}$, $v_1, v_2, v \in V$, and $w_1, w_2, w \in W$.

The image of $(v, w) \in V \times W$ under the quotient map $V * W \to V \otimes W$ is denoted

$$v \otimes w \in V \otimes W$$

and called a *pure tensor*. For any $a \in \mathbb{K}, v_1, v_2, v \in V$, $w_1, w_2, w \in W$ it holds that

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w, \quad v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2, \quad av \otimes w = v \otimes aw = a(v \otimes w).$$

Fact. If V has basis $\{v_i\}_{i \in I}$ and W has basis $\{w_j\}_{j \in J}$ then $\{v_i \otimes w_j\}_{(i,j) \in I \times J}$ is a basis for $V \otimes W$.

The tensor product is associative in the sense that we can identify $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$.

Therefore, we can form iterated tensor products $V^{\otimes 0} = \mathbb{K}$ and $V^{\otimes n} = V \otimes V \otimes \cdots \otimes V$ (n terms).

The *tensor algebra* of a vector space V is defined as $\mathcal{T}V = \bigoplus_{n \geq 0} V^{\otimes n}$.

This is an (associative, unital) algebra with product $xy = x \otimes y$ and unit $1 \in \mathbb{K} = V^{\otimes 0} \subset \mathcal{T}V$.

Notice that if $V = 0$ then $\mathcal{T}V = \mathbb{K}$, and that $\mathcal{T}V$ is commutative if and only if $\dim V \leq 1$.

Any choice of basis for V determines an isomorphism from $\mathcal{T}V$ to a free algebra $\mathbb{K}\langle X_1, X_2, \dots \rangle$.

Some notable quotients of $\mathcal{T}V$:

- The *symmetric algebra* of V is $\mathcal{S}V \stackrel{\text{def}}{=} \mathcal{T}V/\langle v \otimes w - w \otimes v : v, w \in V \rangle$.
- The *exterior algebra* is $\bigwedge V \stackrel{\text{def}}{=} \mathcal{T}V/\langle v \otimes v : v \in V \rangle$.

2 Semisimple representations

In this lecture, we begin a new chapter focusing on some general results about algebra representations.

From now on, we will assume that $\mathbb{K}$ is an algebraically closed field, and that A is a $\mathbb{K}$-algebra.

Definition. A representation of A is *semisimple* (sometimes called *completely reducible*) if it is isomorphic to a direct sum of irreducible representations.

As a general rule in mathematical terminology:

"simple" ≡ "irreducible" and "semisimple" ≡ "(direct) sum of simple objects".

Notation. Suppose V is a left A-module. Often we will say that "V is a representation of A": this just means the representation (ρ, V) where $\rho : A \to \mathsf{End}(V)$ is defined by $\rho(a) : x \mapsto ax$ for $a \in A$ and $x \in V$.

Example (Matrix algebras)**.** Let $A = \mathsf{Mat}_n(\mathbb{K})$ be the algebra of $n \times n$ matrices over $\mathbb{K}$.

Let $V = \mathbb{K}^n$ be the $\mathbb{K}$-vector space of column vectors with n rows.

We can transform any $v \in V$ by multiplying it on the left by a matrix $X \in A$ to get another vector Xv.

This makes V into an A-representation.

This representation is irreducible since if $v, w \in W$ and $v \neq 0$ then some $X \in A$ has $Xv = w$.

Therefore every nonzero vector is *cyclic* in the sense that it is not contained in any proper A-subrepresentation.

We have $\mathsf{End}(V) = A$, which is also an A-representation, via the *(left) regular representation* in which one matrix acts on another by matrix multiplication $X : Y \mapsto XY$.

The regular representation of A is semisimple as we have $A \cong V^{\oplus n}$ as A-representations.

An explicit isomorphism $A \xrightarrow{\sim} V^{\oplus n}$ is the map sending

$$X = \begin{bmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & & \vdots \\ X_{n1} & \cdots & X_{nn} \end{bmatrix} \mapsto \left(\begin{bmatrix} X_{11} \\ \vdots \\ X_{n1} \end{bmatrix}, \begin{bmatrix} X_{12} \\ \vdots \\ X_{n2} \end{bmatrix}, \cdots, \begin{bmatrix} X_{1n} \\ \vdots \\ X_{nn} \end{bmatrix} \right).$$

Notation. Here we define $V^{\oplus n}$ to be the set of n-tuples $(v_1, v_2, \ldots, v_n)$ where each $v_i \in V$ and where

$$(v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots, w_n) \overset{\text{def}}{=} (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n),$$
$$c(v_1, v_2, \ldots, v_n) \overset{\text{def}}{=} (cv_1, cv_2, \ldots, cv_n),$$

for $v_i, w_i \in V$ and $c \in \mathbb{K}$.

Example. Suppose A is any algebra and V is an irreducible A-representation with $\dim(V) = n < \infty$.

Then $\mathsf{End}(V) = \{\text{all linear maps } L : V \to V\}$ is an A-representation for the action

$$a \cdot L : v \mapsto a \cdot L(v) \quad \text{for } a \in A \text{ and } v \in V.$$

This representation is semisimple with $\mathsf{End}(V) \cong V^{\oplus n}$ as A-representations.

If V has basis $\{v_1, \ldots, v_n\}$ then the map $L \mapsto (L(v_1), \cdots, L(v_n))$ is an isomorphism $\mathsf{End}(V) \xrightarrow{\sim} V^{\oplus n}$.

3 Subrepresentation of semisimple representations

Our main results today are derived from the following technical property.

It tells us that all subrepresentations of semisimple representations are semisimple.

Proposition. Let $V_1, \cdots, V_m$ be irreducible A-representations with $\dim(V_i) < \infty$ and $V_i \not\cong V_j$ if $i \neq j$. Consider the A-representation $V = \bigoplus_{i=1}^m V_i^{\oplus n_i}$ where $n_1, n_2, \ldots, n_m$ are nonnegative integers. Now suppose W is a subrepresentation of V. Then:

(1) For some integers $0 \leq r_i \leq n_i$ there is an isomorphism $\phi : \bigoplus_{i=1}^m V_i^{\oplus r_i} \xrightarrow{\sim} W$.

(2) The map $\bigoplus_{i=1}^m V_i^{\oplus r_i} \xrightarrow{\phi} W \hookrightarrow V$ is a direct sum of inclusions $\phi_i : V_i^{\oplus r_i} \hookrightarrow V_i^{\oplus n_i}$ of the form

$$\phi_i(a_1, a_2, \cdots, a_{r_i}) = \begin{bmatrix} a_1 & a_2 & \cdots & a_{r_i} \end{bmatrix} X_i$$

where each X_i is a full rank $r_i \times n_i$ matrix with values in $\mathbb{K}$.

Proof sketch. If $W = 0$ then the proposition is trivial. Assume $W \neq 0$.

We proceed by induction on $n \overset{\text{def}}{=} n_1 + n_2 + \cdots + n_m$.

If $n = 1$ then we must have $0 \neq W = V$ in which case the result is again obvious.

Assume $n > 1$. Since W is finite-dimensional, it has an irreducible subrepresentation P (see HW1).

Observe that $\mathrm{Hom}_A(P, V) = \bigoplus_{i=1}^m \mathrm{Hom}_A(P, V_i)^{\oplus n_i}$. In this equation:

- each term $\mathrm{Hom}_A(P, V_i)$ on the right side is nonzero if and only if $P \cong V_i$ by Schur's lemma;
- the left side is nonzero since it contains inclusion $P \hookrightarrow W \hookrightarrow V$.

Therefore P must be isomorphic to V_i for some i.

The inclusion $V_i \xrightarrow{\sim} P \hookrightarrow V_i^{\oplus n_i} \hookrightarrow V$ must be given by a map of the form

$$v \mapsto (q_1 v, \cdots, q_{n_i} v)$$

for some scalars $q_i \in \mathbb{K}$ that are not all zero. This is because composing this map with each projection

$$(a_1, \cdots, a_{n_i}) \mapsto a_j \in V_i$$

is a morphism of A-representations $V_i \to V_i$, which must be a scalar map by Schur's lemma.

Let $g \in \mathrm{GL}_{n_i}(\mathbb{K}) = \{\text{invertible } n_i \times n_i \text{ matrices}\}$ act on $V_i^{\oplus n_i}$ on the right by the formula

$$g : (v_1, v_2, \cdots, v_{n_i}) \mapsto \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} g$$

while acting on $V_j^{\oplus n_j}$ for $i \neq j$ as the identity. This gives a right action of the general linear group on V. We may choose $g \in \mathrm{GL}_{n_i}(\mathbb{K})$ such that

$$Pg = \{(0, 0, \cdots, 0, v) : v \in V_i\} \subset V_i^{\oplus n_i}.$$

Then $Wg = W' \oplus V_i$ where $V_i = Pg$ and W' is the kernel of projection $Wg \to Pg$, which satisfies

$$W' \subset V_1^{\oplus n_1} \oplus \cdots \oplus V_i^{\oplus (n_i - 1)} \oplus \cdots \oplus V_m^{\oplus n_m}.$$

Now we apply the proposition to W' by induction, and multiply the resulting inclusion by g^{-1}. □

Corollary. Assume the following setup:

- V is an irreducible finite-dimensional representation of A.

- The elements $v_1, v_2, \ldots, v_n \in V$ are linearly independent.
- The elements $w_1, w_2, \ldots, w_n \in V$ are arbitrary.

Then there exists an element $a \in A$ such that $av_i = w_i$ for all $i = 1, 2, \ldots, n$.

Proof. Assume no such element exists. Then the image of A under the map

$$a \mapsto (av_1, \ldots, av_n)$$

is a proper subrepresentation of $V^{\oplus n}$, which we denote by W.

By Proposition 3 we know that $W \cong V^{\oplus m}$ for some $0 \leq m < n$ and there exists an inclusion

$$\phi : V^{\oplus m} \xrightarrow{\sim} W \hookrightarrow V^{\oplus n}$$

of the form $\phi(a_1, a_2, \cdots, a_m) = \begin{bmatrix} a_1 & a_2 & \cdots & a_m \end{bmatrix} X$ where X is a full rank $m \times n$ matrix.

Since $(v_1, v_2, \cdots, v_n) \in W$, we may choose $a_i \in V$ such that $\phi(a_1, a_2, \cdots, a_m) = (v_1, v_2, \cdots, v_n)$.
Also, since $m < n$, there is nonzero vector

$$\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \in \mathbb{K}^n \qquad \text{such that} \qquad X \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = 0.$$

But now

$$\sum_{i=1}^{n} q_i v_i = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \cdots & a_m \end{bmatrix} X \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = 0$$

which contradicts the linear independence of $v_1, \cdots, v_n$. □

Theorem (*Density theorem*)**.** Let (ρ, V) be an irreducible, finite-dimensional A-representation.
Then the map $\rho : A \to \mathsf{End}(V)$ is surjective.
More strongly, suppose (ρ_1, V_1), (ρ_2, V_2), $\ldots (\rho_r, V_r)$ are irreducible finite-dimensional A-representations.
Assume $(\rho_i, V_i) \not\cong (\rho_j, V_j)$ for all $i \neq j$. Then $\bigoplus_{i=1}^r \rho_i : A \to \bigoplus_{i=1}^r \mathsf{End}(V_i)$ is also surjective.

Proof. For the first claim, choose any $L \in \mathsf{End}(V)$ and suppose $v_1, v_2, \ldots, v_n$ is a basis of V.
Set $w_i = L(v_i)$. By the previous corollary, some $a \in A$ has $\rho(a)v_i = w_i$ for all i so $\rho(a) = L$.

The second claim is nontrivial since direct sums of surjective maps are not necessarily surjective.
For example, the direct sum of the identity map with itself $x \mapsto (x, x)$ which is not surjective.
The desired surjectivity property will be a consequence of the second part of the previous proposition.

Let $Y = \bigoplus_{i=1}^r \mathsf{End}(V_i)$. This is a semisimple A-representation as $\mathsf{End}(V_i) \cong V_i^{\oplus d_i}$ where $d_i = \dim(V_i)$.
By the previous proposition, the subrepresentation

$$W = \left(\bigoplus_{i=1}^r \rho_i\right)(A) \subset Y$$

is isomorphic to $\bigoplus_{i=1}^r V_i^{\oplus m_i}$ for some integers $0 \leq m_i \leq d_i$, and there is an inclusion

$$\phi : \bigoplus_{i=1}^r V_i^{\oplus m_i} \xrightarrow{\sim} W \hookrightarrow Y$$

that is given by a direct sum of inclusions $\phi_i : V_i^{\oplus m_i} \hookrightarrow V_i^{\oplus d_i}$.

Since each ρ_i is surjective, the composition of this inclusion with the projection $Y \to \mathsf{End}(V_i)$ is surjective. Hence each ϕ_i is surjective and $m_i = d_i$. This shows that $\bigoplus_i \rho_i$ is surjective. □