1 Review: direct sums and the density theorem

In the last lecture we discussed *semisimple representations*.

These are representations isomorphic to direct sums of irreducible representations.

Notation. If $V_1, V_2, \dots, V_n$ are $\mathbb{K}$-vector spaces, then we view elements of the direct sum

$$V_1 \oplus V_2 \oplus \cdots \oplus V_n$$

either as tuples $(v_1, v_2, \dots, v_n)$ or as row vectors $\begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$ where each $v_i \in V_i$.

Assume A is an algebra defined over an algebraically closed field $\mathbb{K}$.

We proved the following technical result last time:

Proposition. Let $V_1, \cdots, V_m$ be irreducible A-representations with $\dim(V_i) < \infty$ and $V_i \not\cong V_j$ if $i \neq j$.

Consider the A-representation $V = \bigoplus_{i=1}^m V_i^{\oplus n_i}$ where $n_1, n_2, \dots, n_m$ are nonnegative integers.

Now suppose W is a subrepresentation of V. Then:

(1) For some integers $0 \leq r_i \leq n_i$ there is an isomorphism $\phi : \bigoplus_{i=1}^m V_i^{\oplus r_i} \xrightarrow{\sim} W$.

(2) The map $\bigoplus_{i=1}^m V_i^{\oplus r_i} \xrightarrow{\phi} W \hookrightarrow V$ is a direct sum of inclusions $\phi_i : V_i^{\oplus r_i} \hookrightarrow V_i^{\oplus n_i}$ of the form

$$\phi_i(a_1, a_2, \cdots, a_{r_i}) = \begin{bmatrix} a_1 & a_2 & \cdots & a_{r_i} \end{bmatrix} X_i$$

where each X_i is a full rank $r_i \times n_i$ matrix with values in $\mathbb{K}$.

Here are two consequences of this proposition:

- Suppose V is a finite-dimensional A-representation and $v_1, \dots, v_n \in V$ are linearly independent.

  If V is irreducible then the map $a \mapsto (av_1, \dots, av_n)$ is a surjection $A \to V^{\oplus n}$.
- *Density theorem*: Let $(\rho_1, V_1), (\rho_2, V_2), \dots (\rho_r, V_r)$ be irreducible finite-dimensional A-representations.

  If we have $(\rho_i, V_i) \not\cong (\rho_j, V_j)$ for all $i \neq j$ then $\bigoplus_{i=1}^r \rho_i : A \to \bigoplus_{i=1}^r \mathsf{End}(V_i)$ is surjective.

2 Matrix algebras

We have already seen that the algebra of all $n \times n$ matrices over $\mathbb{K}$ has a unique isomorphism class of irreducible representations, corresponding to the space of column vectors $\mathbb{K}^n$.

We can generalize this to block diagonal matrix algebras.

Choose integers $d_1, d_2, \dots, d_r > 0$. Set $n = \sum_{i=1}^r d_i$.

Let $A = \bigoplus_{i=1}^r \mathsf{Mat}_{d_i}(\mathbb{K})$ where we define $\mathsf{Mat}_d(\mathbb{K})$ to be the algebra of $d \times d$ matrices over $\mathbb{K}$.

Identify A with the subalgebra of $\mathsf{Mat}_n(\mathbb{K})$ of all block diagonal matrices with blocks of size $d_i \times d_i$.

The vector space $\mathbb{K}^n$ is automatically an A-representation. We construct a sequence of subrepresentations:

- Let $V_1 \subseteq \mathbb{K}^n$ be the subspace of vectors with zeros outside rows $1, 2, \dots, d_1$
- Let $V_2 \subseteq \mathbb{K}^n$ be the subspace of vectors with zeros outside rows $d_1 + 1, d_1 + 2, \dots, d_1 + d_2$.
- Let $V_3 \subseteq \mathbb{K}^n$ be the subspace of vectors with zeros outside rows $d_1+d_2+1, d_1+d_2+2, \dots, d_1+d_2+d_3$.

- Define $V_4, \dots, V_r$ likewise, so $V_r \subseteq \mathbb{K}^n$ is the subspace of vectors with zeros outside the last d_r rows.

Notice that we have $\dim(V_i) = d_i$.

Theorem. In this setup, each V_i is an irreducible A-representation.
Every finite-dimensional A-representation is isomorphic to a direct sum of zero or more copies of $V_1, V_2, \dots, V_r$.

Before proving this theorem, we introduce another definition.

Definition. Suppose (V, ρ) is an A-representation.
Let V^* be the vector space of all $\mathbb{K}$-linear maps $\lambda : V \to \mathbb{K}$.
Also let $\rho^* : A \to \mathsf{End}(V^*)$ be the linear map defined by

$$\rho^*(a)(\lambda) : x \mapsto \lambda(\rho(a)(x)) \quad \text{for } a \in A \text{ and } \lambda \in V^*.$$

We refer to the pair (V^*, ρ^*) as the *dual* of (V, ρ). It is a representation of the opposite algebra A^{op}.

Fact. For $A = \bigoplus_{i=1}^r \mathsf{Mat}_{d_i}(\mathbb{K}) \subseteq \mathsf{Mat}_n(\mathbb{K})$, the transpose $X \mapsto X^\top$ is an algebra isomorphism $A \cong A^{\mathsf{op}}$.

Given a linear map between vector spaces $L : V \to W$, define $L^* : W^* \to V^*$ by $L^*(f) = f \circ L$.

Fact. If L is injective then L^* is surjective, and if L is surjective then L^* is injective.

Proof of the theorem. If $v, w \in V_i$ are nonzero then we can always find a matrix $M \in A$ with $Mv = w$.
It follows that V_i has no proper subrepesentations so is an irreducible A-representation.

Let X be some finite m-dimensional representation of A where $m < \infty$.
Then X^* is representation of $A^{\mathsf{op}} \cong A$.
In other words, X^* can be viewed as an A-representation for the action

$$a \cdot \lambda : x \mapsto \lambda(a^\top x) \quad \text{for } x \in X,\ \lambda \in X^*,\ a \in A.$$

Choose a basis $\{\lambda_1, \dots, \lambda_m\}$ for X^*. Then let $\phi : A \oplus A \oplus \cdots \oplus A = A^{\oplus m} \to X^*$ be the map

$$\phi(a_1, a_2, \dots, a_m) = a_1\lambda_1 + a_2\lambda_2 + \cdots + a_m\lambda_m.$$

Because $\mathbb{K} \subseteq A$, this map is surjective. Therefore, the dual map $\phi^* : X \to (A^{\oplus m})^*$ is injective.

<u>Key claim</u>: The A-representations $(A^{\oplus m})^*$ and $A^{\oplus m}$ are isomorphic.
If we can prove this, then it will follow that X is isomorphic to a subrepresentation of $A^{\oplus m}$.
Viewing a matrix as a tuple of column vectors gives an isomorphism $A \cong \bigoplus_{i=1}^r V_i^{\oplus d_i}$ as A-representations.
So if we can prove our key claim that it would follow that

$$X \cong \left(\text{a subrepresentation of } A^{\oplus m} \cong \textstyle\bigoplus_{i=1}^r V_i^{\oplus m d_i}\right).$$

By our technical proposition this would imply that $X \cong \bigoplus_{i=1}^r V_i^{\oplus s_i}$ for some integers $s_i \geq 0$ as desired.

It suffices to show the $m = 1$ case of the key claim since $(A^*)^{\oplus m} \cong (A^{\oplus m})^*$.

Let A act on A^* by $a \cdot \lambda : x \mapsto \lambda(a^\top x)$ for $a \in A$ and $\lambda \in A^*$.

Define $\Theta : A \to A^*$ to be the linear map $\Theta : a \mapsto (x \mapsto \mathrm{trace}(a^\top x))$.

Then Θ is a bijection since it is a nonzero linear map with trivial kernel and $\dim(A) = \dim(A^*) < \infty$.

The map Θ also a homomorphism of A-representations since we have

$$\Theta(gh)(x) = \mathrm{trace}(h^\top g^\top x) = \Theta(h)(g^\top x) = (g \cdot \Theta(h))(x) \quad \text{for } g, h, x \in A,$$

which implies that $\Theta(gh) = g \cdot \Theta(h)$. Thus $\Theta : A \xrightarrow{\sim} A^*$ is an isomorphism of A-representations. □

3 Filtrations

Let A be an algebra defined over any field. Suppose V is an A-representation.

Definition. A *filtration* of V is a finite, increasing sequence of subspaces

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$$

where each V_i is subrepresentation of V.

Lemma. If $\dim(V) < \infty$ then V has a filtration with V_i/V_{i-1} an irreducible A-representation if $i > 0$.

Proof. We argue by induction on $\dim V$.

If $\dim V \leq 1$ then the result is trivial: just take $n = 1$ and $V_n = V$.

Assume $\dim V > 1$ and choose any irreducible subrepresentation $V_1 \subset V$.

Then let $U = V/V_1$. By induction there is a filtration

$$0 = U_0 \subset U_1 \subset \cdots \subset U_{n-1} = U$$

in which each quotient U_i/U_{i-1} is irreducible.

Let V_i be the preimage of U_{i-1} under the quotient map $V \to V/V_1 = U$. Then

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

gives the desired filtration, since $V_i/V_{i-1} \simeq (V_i/V_1)/(V_{i-1}/V_1) = U_{i-1}/U_{i-2}$ for $i > 1$. □

4 Radicals of finite-dimensional algebras

Continue to let A be an algebra defined over any field, but now assume that $\dim(A) < \infty$.

Definition. The *radical* of A is the set of elements

$$\mathsf{Rad}(A) = \{a \in A : \rho(a) = 0 \text{ for all irreducible } A\text{-representations } (\rho, V)\}.$$

Proposition. The set $\mathsf{Rad}(A)$ is a two-sided ideal of A.

Proof. The set $\mathsf{Rad}(A)$ is a subspace of A since if (ρ, V) is a representation then

$$\rho(x) = 0 \implies \rho(cx) = c\rho(x) = 0 \quad \text{and} \quad \rho(x) = 0 = \rho(y) \implies \rho(x+y) = \rho(x) + \rho(y) = 0$$

for all $x, y \in A$ and $c \in \mathbb{K}$. The radical of A is a two-sided ideal since if $a, b \in A$ then

$$\rho(x) = 0 \implies \rho(axb) = \rho(a)\rho(x)\rho(b) = 0.$$

□

Let I be a two-sided idea in A. For integers $n \geq 1$, let

$$I^n = \mathbb{K}\text{-span}\{x_1 x_2 \cdots x_n : x_1, x_2, \ldots, x_n \in I\} \subseteq I.$$

We say that I is *nilpotent* if $I^n = 0$ for some $n > 0$.

For example, the subspace of strictly upper triangular matrices is a nilpotent ideal in $\mathsf{Mat}_n(\mathbb{K})$.

Proposition. Suppose I is a nilpotent two-sided ideal in A. Then $I \subseteq \mathsf{Rad}(A)$.

Proof. Choose any irreducible A-representation V and pick a nonzero element $0 \neq v \in V$.

Then the subspace $Iv = \{xv : x \in I\}$ is a subrepresentation, which therefore must be either V or 0.

We cannot have $Iv = V$ as then there is some $x \in I$ with $xv = v$, which is impossible as $x^n = 0$.

Therefore $Iv = 0$. Since V was arbitrary, it follows that $I \subseteq \mathsf{Rad}(A)$. □

The following shows that $\mathsf{Rad}(A)$ is the largest nilpotent two-sided ideal in A.

Proposition. $\mathsf{Rad}(A)$ is itself a nilpotent two-sided ideal in A.

Proof. Since $\dim(A) < \infty$, the previous section gives us a filtration of the regular representation

$$0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = A$$

in which each A_i is a left ideal and each quotient A_i/A_{i-1} is irreducible as an A-representation.

Each $x \in \mathsf{Rad}(A)$ acts as zero on A_i/A_{i-1} and this means that $xA_i \subseteq A_{i-1}$.

Therefore if $x_1, x_2, \ldots, x_n \in \mathsf{Rad}(A)$ then $x_1 x_2 \cdots x_n A \subseteq A_0 = 0$. Hence $\mathsf{Rad}(A)$ is nilpotent. □

5 Representations of finite-dimensional algebras

Now let A be a **finite-dimensional** algebra defined over an **algebraically closed field**.

As a final application, we classify all representations of A.

Theorem. A has finitely many isomorphism classes of irreducible representations $V_1, V_2, \ldots, V_r$ and

$$A/\mathsf{Rad}(A) \cong \textstyle\bigoplus_{i=1}^r \mathsf{End}(V_i)$$

as $\mathbb{K}$-algebras. Moreover, every irreducible A-representation is finite-dimensional.

Notice that since $\dim(V_i)$ is finite, we have $\mathsf{End}(V_i) \cong \mathsf{Mat}_d(\mathbb{K})$ for $d = \dim V_i$.

Therefore $A/\mathsf{Rad}(A)$ is isomorphic to a block diagonal matrix algebra of the form considered earlier today.

Proof. Suppose V is an A-representation.

If $0 \neq x \in V$ then Ax is a nonzero subrepresentation of dimension at most $\dim(A) < \infty$.

Therefore, if V is irreducible then we must have $V = Ax$ and $\dim V \leq \dim(A) < \infty$.

Now suppose (ρ_1, V_1), (ρ_2, V_2), $\dots (\rho_r, V_r)$ are pairwise non-isomorphic, irreducible A-representations.

We know from the density theorem (which requires $\mathbb{K}$ to be algebraically closed) that the direct sum

$$\phi = \bigoplus_{i=1}^r \rho_i : A \to \bigoplus_{i=1}^r \mathsf{End}(V_i)$$

is a surjective map. Since each $\mathsf{End}(V_i)$ has dimension $(\dim V_i)^2$, we have

$$r \leq \sum_{i=1}^r (\dim V_i)^2 \leq \dim A < \infty.$$

Thus the number of isomorphism classes of irreducible A-representations is finite and at most $\dim(A)$.

Finally, assume r is maximal above, so that every irreducible A-representation is isomorphic to some V_i.

Then $\mathsf{Rad}(A) = \ker(\phi)$ so ϕ passes to an isomorphism $A/\mathsf{Rad}(A) \cong \bigoplus_{i=1}^r \mathsf{End}(V_i)$. □

Corollary. If $V_1, V_2, \dots, V_r$ are pairwise non-isomorphic irreducible A-representations then

$$\sum_{i=1}^r (\dim V_i)^2 \leq \dim A.$$