1 Review: representations of finite-dimensional algebras

Let $\mathbb{K}$ be an algebraically closed field.

For integers $n > 0$, let $\mathsf{Mat}_n(\mathbb{K})$ be the algebra of $n \times n$ matrices over $\mathbb{K}$.

Remember that if V is an n-dimensional $\mathbb{K}$-vector space, then $\mathsf{End}(V) \cong \mathsf{Mat}_n(\mathbb{K})$.

Theorem. Suppose $A = \bigoplus_{i=1}^r \mathsf{Mat}_{d_i}(\mathbb{K})$ for some integers $d_1, d_2, \dots, d_r > 0$.

Then A has an irreducible representation V_i of dimension d_i for $i \in \{1, 2, \dots, r\}$, with $V_i \not\cong V_j$ if $i \neq j$.

Every finite-dimensional representation of A is a direct sum of copies of $V_1, V_2, \dots, V_r$.

If we view $A \subseteq \mathsf{Mat}_n(\mathbb{K})$ as a subalgebra of block diagonal $n \times n$ matrices where $n = d_1 + d_2 + \cdots + d_r$, then we can construct V_i as the subspace of vectors in $\mathbb{K}^n$ with zeros outside the rows indexed by

$$(d_1 + d_2 + \cdots + d_{i-1}) + \{1, 2, \dots, d_i\}.$$

Now suppose $\dim(A) < \infty$. The radical of A is

$$\begin{aligned}\mathsf{Rad}(A) &\overset{\text{def}}{=} \{\text{the elements in } A \text{ that act as zero in every irreducible representation of } A\} \\ &= (\text{the largest nilpotent 2-sided ideal in } A).\end{aligned}$$

Theorem. An algebra A that is finite-dimensional and defined over an algebraically closed field has finitely many irreducible representations $V_1, \dots, V_r$ up to isomorphism.

Each of these representations is of finite dimension $d_i = \dim(V_i)$ and it holds that

$$A/\mathsf{Rad}(A) \cong \textstyle\bigoplus_{i=1}^r \mathsf{End}(V_i) \cong \bigoplus_{i=1}^r \mathsf{Mat}_{d_i}(\mathbb{K}).$$

Since each $\mathsf{End}(V_i)$ has dimension $d_i^2 = \dim(V_i)^2$, it follows that

$$\dim(A) - \dim(\mathsf{Rad}(A)) = \textstyle\sum_{i=1}^r \dim(V_i)^2 \leq \dim(A).$$

Example. Suppose $A = \mathbb{K}[x]/(x^n)$ where $n \geq 1$. Then we can view A as the vector space

$$A = \mathbb{K}\text{-span}\{1, x, \dots, x^{n-1}\}.$$

Let (ρ, V) is a finite-dimensional representation of A. Then

$$\begin{aligned} x^n = 0 \text{ in } A \quad &\implies \quad V \text{ has a basis in which the matrix of } \rho(x) \text{ is strictly upper-triangular} \\ &\implies \quad \text{if } V \text{ is irreducible then } \rho(x) = 0 \text{ and } \dim V = 1. \end{aligned}$$

Thus $A/\mathsf{Rad}(A) \cong \mathsf{End}(\mathbb{K}) = \mathbb{K}$.

Note that we can see directly that $\mathsf{Rad}(A) = (x)$ as this is the largest nilpotent two-sided ideal in A.

Example. Suppose A is the subalgebra of upper-triangular matrices in $\mathsf{Mat}_n(\mathbb{K})$.

Let (ρ_i, V_i) be the representation of A in which $V_i = \mathbb{K}$ and $\rho_i : A \to \mathsf{End}(\mathbb{K}) = \mathbb{K}$ sends $a \in A$ to

$$\rho_i(a) = a_{ii} \quad (\text{the diagonal entry of } a \text{ in row } i)$$

There representations for $i = 1, 2, \dots, n$ are irreducible and pairwise-non-isomorphic.

They give all irreducible representations of A (up to isomorphism) since

$$\mathsf{Rad}(A) = \{\text{strictly upper-triangular matrices in } \mathsf{Mat}_n(\mathbb{K})\},$$

as this is the largest nilpotent two-sided ideal in A, and so

$$A/\mathsf{Rad}(A) \cong \mathbb{K}^n \implies \text{there are exactly } n \text{ isomorphism classes of irreducible } A\text{-representations}.$$

2 Semisimple algebras

Our main new results today concern the following class of algebras.

Definition. A finite-dimensional algebra A is called *semisimple* if $\mathsf{Rad}(A) = 0$.

Recall that a representation is *semisimple* if it is a direct sum of irreducible representations.

Assume A is an algebra over an algebraically closed field $\mathbb{K}$ with $\dim(A) < \infty$.

Proposition. The following properties are equivalent:

(1) A is semisimple.

(2) $\dim(A) = \sum_{i=1}^r \dim(V_i)^2$ where $V_1, \dots, V_r$ are the isomorphism classes of irreducible A-representations.

(3) $A \cong \bigoplus_{i=1}^r \mathsf{Mat}_{d_i}(\mathbb{K})$ for some list of positive integers $d_1, d_2, \dots, d_r$.

(4) Any finite-dimensional representation of A is semisimple.

(5) The regular representation of A is semisimple.

Proof. The equivalence of the first three properties follows from the previous theorem.

We now argue that (3) $\implies$ (4) $\implies$ (5) $\implies$ (3).

The implication (3) $\implies$ (4) holds by the first theorem on the previous page.

The implication (4) $\implies$ (5) is trivial.

To show that (5) $\implies$ (3), assume the regular representation of A is semisimple.

Then we can write $A = \bigoplus_{i=1}^r d_i V_i$ where $V_1, V_2, \dots, V_r$ are irreducible and pairwise-non-isomorphic.

Here each V_i is an A-representation and we write $d_i V = V^{\oplus d_i}$.

Now consider $\mathsf{End}_A(A) = \{\text{morphisms } A \to A \text{ as } A\text{-representations}\} = \mathrm{Hom}_A(A, A)$.

Schur's lemma (which requires $\mathbb{K}$ to be algebraically closed) tells us that

- $\mathsf{End}_A(V_i) = \mathbb{K}$ so $\mathsf{End}_A(d_i V_i) \cong \mathsf{Mat}_{d_i}(\mathbb{K})$, and
- $\mathrm{Hom}_A(V_i, V_j) = 0$ if $i \neq j$, so $\mathrm{Hom}_A(d_i V_i, d_j V_j) = 0$ if $i \neq j$.

Thus, we compute $\mathsf{End}_A(A) = \mathrm{Hom}_A(A, A) = \bigoplus_{i,j} \mathrm{Hom}(d_i V_i, d_j V_j) \cong \bigoplus_i \mathsf{Mat}_{d_i}(\mathbb{K})$.

Exercise: Show that $(\mathsf{End}_A(A))^{\mathsf{op}} \cong A$ or equivalently that $\mathsf{End}_A(A) \cong A^{\mathsf{op}}$.

Last time: There is an isomorphism $(\bigoplus_i \mathsf{Mat}_{d_i}(\mathbb{K}))^{\mathsf{op}} \cong \bigoplus_i \mathsf{Mat}_{d_i}(\mathbb{K})$ afforded by the transpose map.

Thus we have $A \cong (\mathsf{End}_A(A))^{\mathsf{op}} \cong (\bigoplus_i \mathsf{Mat}_{d_i}(\mathbb{K}))^{\mathsf{op}} \cong \bigoplus_i \mathsf{Mat}_{d_i}(\mathbb{K})$.

This is property (3), so (5) $\implies$ (3) as desired. □

3 Characters

Let A be an algebra defined over an algebraically closed field $\mathbb{K}$.

Suppose (ρ, V) is a finite-dimensional representation of A.

Definition. The *character* of (ρ, V) is the linear map $\chi_{(\rho,V)} : A \to \mathbb{K}$ with the formula

$$\chi_{(\rho,V)}(a) = \mathrm{trace}(\rho(a)) \quad \text{for } a \in A.$$

How can we compute the trace of $\phi \in \mathsf{End}(V)$?

First choose a basis $e_1, e_2, \dots, e_n$ of V. Then $\mathrm{trace}(\phi) = \sum_{i=1}^n (\text{coefficient of } e_i \text{ in } \phi(e_i))$.

Some well-known facts about traces:

(1) The method just given to compute the trace does not depend on the choice of basis.

(2) We have $\mathrm{trace}(\phi_1\phi_2) = \mathrm{trace}(\phi_2\phi_1)$ for all $\phi_1, \phi_2 \in \mathsf{End}(V)$.

Therefore $\mathrm{trace}(\phi_1\phi_2\phi_1^{-1}) = \mathrm{trace}(\phi_2)$ if ϕ_1 is invertible.

(3) If $(\rho_1, V_1) \cong (\rho_2, V_2)$ are finite-dimensional A-representations then $\chi_{(\rho_1,V_1)} = \chi_{(\rho_2,V_2)}$.

To abbreviate, we will sometimes write χ_V instead of $\chi_{(\rho,V)}$.

Let $[A, A] = \mathbb{K}\text{-span}\left\{[a, b] \overset{\text{def}}{=} ab - ba : a, b \in A\right\}$. We view this as just a vector space.

Fact. We always have $[A, A] \subseteq \ker(\chi_{(\rho,V)})$

Proof. Let $\chi = \chi_{(\rho,V)}$. Then

$$\chi(ab - ba) = \mathrm{trace}(\rho(ab)) - \mathrm{trace}(\rho(ba)) = \mathrm{trace}(\rho(a)\rho(b)) - \mathrm{trace}(\rho(b)\rho(a)) = 0.$$

□

In the following theorem, $\dim(A)$ is not required to be finite.

Theorem. The characters of any list of non-isomorphic irreducible finite-dimensional A-representations are linearly independent (and, in particular, are distinct).

Proof. Suppose $(\rho_1, V_1), (\rho_2, V_2), \dots, (\rho_r, V_r)$ are pairwise non-isomorphic irreducible finite-dimensional A-representations. Let $\chi_i = \chi_{(\rho_i,V_i)}$.

By the density theorem (which requires $\mathbb{K}$ to be algebraically closed), the map

$$\rho_1 \oplus \cdots \oplus \rho_r : A \to \mathsf{End}(V_1) \oplus \cdots \oplus \mathsf{End}(V_r)$$

is surjective. Therefore, if $\sum_{i=1}^r \lambda_i \chi_i = 0$ for some coefficients $\lambda_1, \lambda_2, \dots, \lambda_r \in \mathbb{K}$, then

$$\textstyle\sum_{i=1}^r \lambda_i \mathrm{trace}(M_i) = 0 \quad \text{for any } M_i \in \mathsf{End}(V_i) \text{ chosen independently,}$$

which is only possible if $\lambda_1 = \lambda_2 = \cdots = \lambda_r = 0$. □

We say that a character $\chi_{(\rho,V)}$ is *irreducible* if (ρ, V) is irreducible.

Theorem. Assume A is semisimple and $\dim(A) < \infty$.

The irreducible characters of A are a basis for the vector space $(A/[A,A])^*$ of linear maps $A/[A,A] \to \mathbb{K}$.

Proof. Each character χ has $[A,A] \subset \ker(\chi)$, so χ belongs to $(A/[A,A])^*$.

Since A is semisimple we may assume $A = \mathsf{Mat}_{d_1}(\mathbb{K}) \oplus \cdots \oplus \mathsf{Mat}_{d_r}(\mathbb{K})$.

Then $[A,A] = \bigoplus_{i=1}^r [\mathsf{Mat}_{d_i}(\mathbb{K}), \mathsf{Mat}_{d_i}(\mathbb{K})]$.

We claim that $[\mathsf{Mat}_d(\mathbb{K}), \mathsf{Mat}_d(\mathbb{K})] = \mathfrak{sl}_d(\mathbb{K})$, where $\mathfrak{sl}_d(\mathbb{K}) \overset{\text{def}}{=} \{M \in \mathsf{Mat}_n(\mathbb{K}) : \mathrm{trace}(M) = 0\}$

To prove the claim, note that the trace map certainly vanishes on $[\mathsf{Mat}_d(\mathbb{K}), \mathsf{Mat}_d(\mathbb{K})]$.

In addition, $\mathfrak{sl}_d(\mathbb{K})$ is spanned by the elements

$$E_{ij} = [E_{ik}, E_{kj}] \text{ for } i \neq j \quad \text{and} \quad E_{ii} - E_{i+1,i+1} = [E_{i,i+1}, E_{i+1,i}]$$

where E_{ij} is the $n \times n$ *elementary matrix* with 1 in position (i,j) and 0 elsewhere.

The claim implies we have $A/[A,A] \cong \mathbb{K}^r$ since $\mathsf{Mat}_d(\mathbb{K})/\mathfrak{sl}_d(\mathbb{K}) \cong \mathbb{K}$.

Finally, we know that A has r distinct irreducible characters.

These are linearly independent elements of $(A/[A,A])^*$.

Hence they must be a basis as $\dim(A/[A,A])^* = \dim(A/[A,A]) = r$. □

4 Two general results

We finish today with two general results that can be applied to all finite-dimensional algebras.

Assume $\dim(A) < \infty$ and that the ambient field $\mathbb{K}$ is algebraically closed.

Let V be a finite-dimensional representation of A.

Theorem (*Jordan-Hölder theorem*)**.** Suppose we have filtrations

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V \quad \text{and} \quad 0 = V_0' \subset V_1' \subset \cdots \subset V_m' = V$$

where V_i and V_i' are subrepresentations such that the quotients

$$W_i \overset{\text{def}}{=} V_i/V_{i-1} \qquad \text{and} \qquad W_i' \overset{\text{def}}{=} V_i'/V_{i-1}'$$

are irreducible. Then $m = n$ and there exists a permutation σ such that $W_{\sigma(i)} \cong W_i'$ for all i.

We call the common length $m = n$ of these filtrations the *length* of the representation V.

Proof. We can give a simple proof when $\mathrm{char}(\mathbb{K}) = 0$. In this case, it follows by a homework exercise that $\chi_V = \chi_W + \chi_{V/W}$ if A is any subrepresentation of V, and so we have $\chi_V = \sum_{i=1}^n \chi_{W_i} = \sum_{i=1}^m \chi_{W_i'}$.

Then we can deduce the theorem by the linear independence of the irreducible characters of A.

This argument does not work for $\mathrm{char}(\mathbb{K}) = p > 0$, as the multiplicities of the irreducible characters in the decomposition of χ_V could be multiples of p. See the textbook for the argument handling this case. □

We maintain the same setup for A and V in the next theorem.

Theorem (*Krull-Schmidt theorem*)**.** There is a decomposition of V, which is unique up to isomorphism and rearrangement of factors, as a direct sum of indecomposable A-representations.

The existence of such a decomposition follows fairly easily by induction on $\dim(V)$.

The uniqueness claim in the theorem is harder to show.

We will give the complete proof of the theorem next time.