1 Review: semisimple algebras and characters

Let $\mathbb{K}$ be an algebraically closed field. Suppose A is a finite-dimensional algebra over $\mathbb{K}$.

Then every irreducible A-representation V has $\dim V < \infty$ since if $0 \neq x \in V$ then $Ax = V$ but

$$\dim(Ax) \leq \dim(A) < \infty.$$

Recall that an A-representation is *semisimple* if it is a direct sum of irreducible subrepresentations.

The algebra A is *semisimple* if any (and hence all) of the following equivalent properties hold:

(1) $\mathsf{Rad}(A) \overset{\text{def}}{=} \{$elements in A that act as zero on every irreducible A-representation$\}$ is zero.

(2) If $V_1, V_2, \dots, V_r$ represent all distinct isomorphism classes of irreducible A-representations, then

$$\dim(A) = \sum_{i=1}^r \dim(V_i)^2.$$

(3) A is isomorphic to a finite direct sum of matrix algebras $\mathsf{Mat}_{d_1}(\mathbb{K}) \oplus \mathsf{Mat}_{d_2}(\mathbb{K}) \oplus \cdots \oplus \mathsf{Mat}_{d_r}(\mathbb{K})$.

(4) Every A-representation of finite dimension is semisimple.

(5) The regular representation of A is semisimple.

Let A be any $\mathbb{K}$-algebra (not necessarily of finite dimension).

Assume (ρ, V) is an A-representation with $\dim(V) < \infty$.

The *character* of (ρ, V) is the linear map $\chi_{(\rho,V)} : A \to \mathbb{K}$ with the formula

$$\chi_{(\rho,V)}(a) = \mathrm{trace}(\rho(a)) \overset{\text{def}}{=} \textstyle\sum_{i=1}^n (\text{coefficient of } b_i \text{ in } \rho(a)(b_i)) \quad \text{for any basis } b_1, b_2, \dots, b_n \text{ of } V.$$

Fact. If (ρ, V) and (ρ', V') are isomorphic finite-dimensional A-representations then $\chi_{(\rho,V)} = \chi_{(\rho',V')}$.

We say that $\chi_{(\rho,V)}$ is *irreducible* when (ρ, V) is irreducible.

Theorem. The characters of non-isomorphic irreducible finite-dimensional A-representations are linearly independent (and therefore distinct).

Fact. It always holds that $\mathsf{kernel}(\chi_{(\rho,V)}) \supset [A, A] \overset{\text{def}}{=} \mathbb{K}\text{-span}\{ab - ba : a, b \in A\}$

This means we can view a character as a linear map $A/[A, A] \to \mathbb{K}$.

Theorem. If A is finite-dimensional and semisimple, then the irreducible characters of A are a basis for the dual space $(A/[A, A])^*$. If $\mathrm{char}(\mathbb{K}) = 0$, then two finite-dimensional A-representations are isomorphic if and only if they have same characters.

2 Two general theorems

Our goal today is to establish two general theorems about representations of an algebra A that is not necessarily semisimple. We proved the first of these theorems last time:

Theorem (*Jordan-Hölder theorem*)**.** If V is an A-representation with $\dim(V) < \infty$ then there exists a filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ where each V_i is a subrepresentation and each quotient V_i/V_{i-1} is irreducible. Moreover, any other filtration with these properties has same length n and the same irreducible quotients up to isomorphism and permutations of indices.

Today we will supply the proof of the next theorem:

Theorem (*Krull-Schmidt theorem*)**.** If V is an A-representation with $\dim(V) < \infty$ then there exists a decomposition $V = \bigoplus_{i\in I} V_i$ where each V_i is an indecomposable subrepresentation, and this decomposition is unique up to isomorphism and rearrangement of factors.

Remember that when A is semisimple, every indecomposable representation is irreducible, but for a general algebra we may not be able to decompose a representation into a direct sum of irreducible subrepresentations. The Krull-Schmidt theorem is relevant to the latter setting.

We will prove the Krull-Schmidt theorem after establishing a few lemmas.

A linear map $\theta : W \to W$ is *nilpotent* if $\theta^N \stackrel{\text{def}}{=} \theta \circ \theta \circ \cdots \circ \theta$ is zero for some $N > 0$.

Lemma. Let W be an indecomposable A-representation where $\dim W < \infty$.

Suppose $\theta : W \to W$ is a morphism of A-representations. Then θ is either an isomorphism or nilpotent.

Proof. For $\lambda \in \mathbb{K}$, the *generalized λ-eigenspace* of θ is

$$W_\lambda \stackrel{\text{def}}{=} \{x \in W : (\theta - \lambda)^N(x) = 0 \text{ for some } N > 0\}.$$

The subspace W_λ is nonzero if and only if λ is an eigenvalue of θ.

By standard linear algebra we have $W = \bigoplus_\lambda W_\lambda$ where the direct sum is over the eigenvalues of θ.

Observe that each W_λ is an A-subrepresentation.

Since W is indecomposable, θ must only have one eigenvalue λ. If $\lambda = 0$ then θ is nilpotent since $W = W_\lambda$.

If $\lambda \neq 0$ then θ is invertible, and hence an isomorphism of A-representations. □

Lemma. Let W be an indecomposable A-representation where $\dim(W) < \infty$.

Suppose $\theta_s : W \to W$ for $s = 1, 2, \ldots, n$ are nilpotent morphisms of A-representations.

Then the sum $\theta = \theta_1 + \cdots + \theta_n$ is also nilpotent.

Proof. We argue by contradiction. Let n be minimal such that the lemma fails.

Then we must have $n > 1$ and θ is not nilpotent. Hence θ is invertible by previous lemma.

Therefore we can write $1 = \theta^{-1}\theta = \sum_{s=1}^n \theta^{-1}\theta_s$.

Since $\mathsf{kernel}(\theta^{-1}\theta_s) = \theta^{-1}(\mathsf{kernel}(\theta_s)) \neq 0$, each $\theta^{-1}\theta_s$ is not invertible, hence nilpotent by the lemma.

But then $1 - \theta^{-1}\theta_n = \sum_{s=1}^{n-1} \theta^{-1}\theta_s$ is invertible, and therefore not nilpotent, since if X is nilpotent then

$$(1 - X)^{-1} = 1 + X + X^2 + \ldots.$$

This contradicts the minimality of n, so we conclude that the lemma actually holds for all n. □

Proof of the Krull-Schmidt theorem. To show the existence of an indecomposable decomposition

$$V = \bigoplus_{i\in I} V_i$$

note that if V is not indecomposable then must exist nonzero subrepresentations U and W with

$$V = U \oplus W,$$

and by induction on dimension we can assume that U and W have indecomposable decompositions.

The hard part is showing the uniqueness of the resulting decomposition.

Suppose $V = \bigoplus_{s=1}^m V_s = \bigoplus_{s=1}^n W_s$ where each V_s and W_s is an indecomposable subrepresentation. Let

$$\begin{array}{c} i_s : V_s \hookrightarrow V \\ j_s : W_s \hookrightarrow V \end{array} \quad\text{and}\quad \begin{array}{c} p_s : V \twoheadrightarrow V_s \\ q_s : V \twoheadrightarrow W_s \end{array}$$

be the natural inclusion and projection maps.

Define $\theta_s = p_1 \circ j_s \circ q_s \circ i_1$ so that

$$\theta_s : V_1 \xhookrightarrow{i_1} V \xrightarrow{q_s} W_s \xhookrightarrow{j_s} V \xrightarrow{p_1} V_1.$$

Note that i_s, p_s, j_s, q_s, and θ_s are all morphisms of A-representations.

Also, notice that the sum $\theta_1 + \theta_2 + \cdots + \theta_n$ is the identity map $V_1 \to V_1$.

Each θ_s is either nilpotent or an isomorphism by our first lemma.

Since $\sum_{s=1}^n \theta_s$ is not nilpotent, some θ_s is an isomorphism by our second lemma.

Without loss of generality we can assume that $\theta_1 : V_1 \to V_1$ is an isomorphism. Since

$$\theta_1 : V_1 \xrightarrow{q_1\circ i_1} W_1 \xrightarrow{p_1\circ j_1} V_1$$

is an isomorphism, we must have $W_1 = \text{image}(q_1 \circ i_1) \oplus \mathsf{kernel}(p_1 \circ j_1)$.

As W_1 is indecomposable, both $p_1 \circ j_1 : W_1 \to V_1$ and $q_1 \circ i_1 : V_1 \to W_1$ must be isomorphisms.

Let $V' = \bigoplus_{s=2}^m V_s$ and $W' = \bigoplus_{s=2}^n W_s$ so that $V = V_1 \oplus V' = W_1 \oplus W'$. Let

$$h : V' \hookrightarrow V \twoheadrightarrow W'$$

be the composition of the obvious inclusion and projection maps.

Clearly $\mathsf{kernel}(h) = V' \cap W_1$, but $(p_1 \circ j_1)(V' \cap W_1) = 0$.

Since $p_1 \circ j_1 : W_1 \to V_1$ is isomorphism, must have $\mathsf{kernel}(h) = 0$ so $h : V' \to W'$ is isomorphism.

Now by induction applied to the decompositions

$$V' = \bigoplus_{s=2}^m V_s \cong \bigoplus_{s=2}^n W_s = W', \tag{1}$$

we must have $m = n$ and there must exist a permutation σ with $V_s \cong W_{\sigma(s)}$ for all s.

This establishes that the same holds for our starting decompositions $V = \bigoplus_{s=1}^m V_s = \bigoplus_{s=1}^n W_s$. □

3 Tensor products of algebras and representations

To finish today's lecture, we briefly discuss representations of tensor product algebras.

Let A and B be $\mathbb{K}$-algebras and write $\otimes = \otimes_{\mathbb{K}}$ for the tensor product for $\mathbb{K}$-vector spaces.

Since A and B are vector spaces, we can consider the vector space $A \otimes B$. It has more structure:

Fact. The vector space $A \otimes B$ is itself a $\mathbb{K}$-algebra for the product given by the bilinear operation

$$(A \otimes B) \times (A \otimes B) \to A \otimes B$$

satisfying $(a \otimes b)(a' \otimes b') \stackrel{\text{def}}{=} aa' \otimes bb'$ for $a, a' \in A$, $b, b' \in B$. The unit for this product is $1_A \otimes 1_B$.

Let V be an A-representation and let W be a B-representation.

Then $V \otimes W$ has a unique structure as an $A \otimes B$-representation in which

$$(a \otimes b)(v \otimes w) \stackrel{\text{def}}{=} av \otimes bw \qquad \text{for } a \in A,\, b \in B,\, v \in V, \text{ and } w \in W.$$

Theorem. Assume $\dim(V) < \infty$ and $\dim(W) < \infty$.

Then $V \otimes W$ is irreducible (as an $A \otimes B$-representation) if V and W are both irreducible.

Proof. Assume V and W are both irreducible and of finite dimension.

By the density theorem, we have surjective maps $\rho_V : A \to \mathsf{End}(V)$ and $\rho_W : A \to \mathsf{End}(W)$.

By general properties of tensor product, the map $\rho_V \otimes \rho_W : A \otimes B \to \mathsf{End}(V) \otimes \mathsf{End}(W)$ is also surjective.

If $\dim(V) < \infty$ and $\dim(W) < \infty$ then there is an isomorphism $\mathsf{End}(V) \otimes \mathsf{End}(W) \cong \mathsf{End}(V \otimes W)$.

But the map $\rho_{V \otimes W} : A \otimes B \to \mathsf{End}(V \otimes W)$ is thus surjective as it is the composition

$$A \otimes B \xrightarrow{\rho_V \otimes \rho_W} \mathsf{End}(V) \otimes \mathsf{End}(W) \xrightarrow{\cong} \mathsf{End}(V \otimes W).$$

Hence $V \otimes W$ is irreducible, since $\rho_{V \otimes W}$ being surjective implies that every $0 \neq x \in V \otimes W$ is cyclic.

(A vector in a representation is *cyclic* if no proper subrepresentation contains it.) □

The previous theorem has a converse.

Theorem. Suppose M is an irreducible $A \otimes B$-representation of finite dimension.

Then $M \cong V \otimes W$ for some irreducible A-representation V and irreducible B-representation W.

Proof sketch. We can assume A and B are finite-dimensional by replacing each algebra by its image under

$$A \hookrightarrow A \otimes B \twoheadrightarrow \mathsf{End}(M) \quad \text{and} \quad B \hookrightarrow A \otimes B \twoheadrightarrow \mathsf{End}(M)$$

where the inclusions send $a \mapsto a \otimes 1_B$ and $b \mapsto 1_A \otimes b$. Next, check that

$$\mathsf{Rad}(A \otimes B) = \mathsf{Rad}(A) \otimes B + A \otimes \mathsf{Rad}(B)$$

so we have

$$(A \otimes B)/\mathsf{Rad}(A \otimes B) \cong A/\mathsf{Rad}(A) \otimes B/\mathsf{Rad}(B)$$

and M is an irreducible representation of this quotient.

Finally, the result can be deduced by identifying the quotient algebras $A/\mathsf{Rad}(A)$ and $B/\mathsf{Rad}(B)$ with explicit (direct sums of) matrix algebras, using the classification of irreducible representations for such algebras and the homework exercise checking that $\mathsf{Mat}_n(\mathbb{K}) \otimes \mathsf{Mat}_m(\mathbb{K}) \cong \mathsf{Mat}_{mn}(\mathbb{K})$. $\square$