1 Review: two structure theorems and tensor products

Let A be an algebra over an algebraically closed field $\mathbb{K}$.

Suppose V is an A-representation with $\dim(V) < \infty$. We have seen general theorems concerning V:

Theorem (*Jordan-Hölder theorem*)**.** There exists a filtration of subrepresentations

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

with irreducible quotients V_i/V_{i-1}, and the length n of the filtration and the isomorphism classes of the quotients are uniquely determined up to permutation of indices.

Theorem (*Krull-Schmidt theorem*)**.** There is a direct sum decomposition of V into indecomposable subrepresentations that is unique up to isomorphism and rearrangement of factors.

Remark. In both theorems, the algebra A can have infinite dimension as long as $\dim(V) < \infty$, since the relevant statements hold for V viewed as an A-representation if and only if they hold for V viewed as a representation of the finite-dimensional algebra $\rho_V(A) \subseteq \mathsf{End}(V)$.

More strongly, one can show that the results even hold when the field $\mathbb{K}$ is not algebraically closed.

However, our proofs from last time do not handle this case.

We also discussed the representations of the tensor product of two $\mathbb{K}$-algebras A and B.

The vector space $A \otimes B$ is also a $\mathbb{K}$-algebra for the product defined on pure tensors by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) \overset{\text{def}}{=} a_1a_2 \otimes b_1b_2 \qquad \text{for } a_1, a_2 \in A \text{ and } b_1, b_2 \in B.$$

If V is an A-representation and W is a B-representation then $V \otimes W$ is an $A \otimes B$-representation with

$$(a \otimes b) \cdot (v \otimes w) \overset{\text{def}}{=} av \otimes bw \quad \text{for } a \in A,\ b \in B,\ v \in V,\ w \in W.$$

Theorem. Suppose V is an A-representation and W is a B-representation. If V and W are both irreducible and finite-dimensional then so is $V \otimes W$ (viewed as an $A \otimes B$-representation). Moreover, up to isomorphism, all irreducible finite-dimensional representations of $A \otimes B$ arise in this way.

We reduced the proof of this theorem to when A and B are semisimple and finite-dimensional.

In this case each algebra is isomorphic to a direct sum of matrix algebras.

In this concrete setting, the theorem is easy to check directly:

Up to isomorphism, $\mathsf{Mat}_m(\mathbb{K})$ and $\mathsf{Mat}_n(\mathbb{K})$ have unique irreducible representations $\mathbb{K}^m$ and $\mathbb{K}^n$.

Hence $\mathbb{K}^m \otimes \mathbb{K}^n \cong \mathbb{K}^{mn}$ is the unique irreducible representation of $\mathsf{Mat}_m(\mathbb{K}) \otimes \mathsf{Mat}_n(\mathbb{K}) \cong \mathsf{Mat}_{mn}(\mathbb{K})$.

2 Finite group representations

For a $\mathbb{K}$-vector space V let $\mathrm{GL}(V)$ be the group of invertible linear maps $V \to V$.

Let G be a group.

A *representation* of G is a pair (ρ, V) where V is a vector space and $\rho : G \to \mathrm{GL}(V)$ is a homomorphism.

Representations of G are equivalent to representations of the group algebra $\mathbb{K}[G] = \mathbb{K}\text{-span}\{a_g : g \in G\}$.

We think of elements of $\mathbb{K}[G]$ as formal (finite) linear combinations of group elements, writing

$$\textstyle\sum_{g\in G} c_g g \qquad \text{and} \qquad \text{instead of} \qquad \sum_{g\in G} c_g a_g$$

where $c_g \in \mathbb{K}$ and a_g is the formal symbol indexed by $g \in G$.

We are interested in representations of **finite** groups G. In this case $\mathbb{K}[G]$ has finite dimension.

Our first important question to answer is: when is $\mathbb{K}[G]$ semisimple?

From this point on, assume that the group G is finite. Write $|G|$ for its number of elements.

Theorem (*Maschke's theorem*)**.** Assume that $\mathrm{char}(\mathbb{K})$ does not divide $|G|$. Then $\mathbb{K}[G]$ is semisimple.

Proof. Let (ρ, V) be a finite-dimensional G-representation, and hence also a $\mathbb{K}[G]$-representation.

It suffices to check that V is a direct sum of irreducible subrepresentations.

This clearly holds if (ρ, V) is irreducible so assume this is not the case.

Then V must have an irreducible subrepresentation W by one of our homework exercises.

By induction $\dim(V)$, it is enough to show that V has a nonzero subrepresentation U with $V = W \oplus U$.

We can find a subspace $\widetilde{U}$, not necessarily a subrepresentation, with $V = W \oplus \widetilde{U}$ as vector spaces.

Just choose a basis $w_1, w_2, \dots, w_m$ of W, extend this to a basis $w_1, \dots, w_m, u_1, \dots, u_n$ for V, and set

$$\widetilde{U} = \mathbb{K}\text{-span}\{u_1, \dots, u_n\}.$$

Here is the key idea to the proof.

Let $\pi : V \to W$ be linear map with $\pi(w_i) = w_i$ for all i and $\pi(u_j) = 0$ for all j. Then define

$$\sigma = \frac{1}{|G|} \sum_{g\in G} \rho(g) \circ \pi \circ \rho(g^{-1})$$

Finally consider $U = \mathsf{kernel}(\sigma)$. We claim that:

(1) U is a subrepresentation.

(2) $V = W \oplus U$.

Property (1) holds because for any $h \in G$ we have

$$\sigma\rho(h) = \frac{1}{|G|} \sum_{g\in G} \rho(g) \circ \pi \circ \rho(g^{-1}h) = \frac{1}{|G|} \sum_{x\in G} \rho(hx) \circ \pi \circ \rho(x^{-1}) = \rho(h)\sigma,$$

making the substitution $x = h^{-1}g$ in the second equality.

Thus $\sigma(u) = 0$ if and only if $\sigma\rho(h)(u) = \rho(h)\sigma(u) = 0$ for any $h \in G$ and $u \in U$, as $\rho(h)$ is invertible.

For property (2), note that $\rho(W) \subseteq W$ and $\pi(w) = w$ for all $w \in W$, so $\sigma(w) = w$ for all $w \in W$.

Since $\sigma(V) \subseteq W$, it follows that $\sigma^2 = \sigma$. Thus any $v \in V$ can be written as

$$v = \sigma(v) + (v - \sigma(v))$$

where $\sigma(v) \in W$ and $(v - \sigma(v)) \in U$, and we have $W \cap U = 0$ since if $x \in W \cap U$ then $x = \sigma(x) = 0$.

Thus $V = W \oplus U$ as needed. □

Corollary. Assume $\mathrm{char}(\mathbb{K})$ does not divide $|G|$. Then there are finitely many isomorphism classes of irreducible G-representations $\{(\rho_i, V_i)\}_{i\in I}$, all of which have finite dimension, and we have

$$|G| = \sum_{i\in I} \dim(V_i)^2 \quad \text{and} \quad \mathbb{K}[G] \cong \bigoplus_{i\in I} \mathsf{End}(V_i).$$

The representation theory of finite-dimensional semisimple algebras is trivial in the sense that everything is just a direct sum of matrix algebras. What makes the representation theory of finite groups more interesting is the distinguished basis of $\mathbb{K}[G]$ provided by G itself.

Going from this basis to the natural bases of $\mathbb{K}[G]$ viewed as a sum of matrix algebras is complicated.

It turns out that the converse to Maschke's theorem is also true.

Theorem. If $\mathbb{K}[G]$ is semisimple then $\mathrm{char}(\mathbb{K})$ does not divide $|G|$.

Proof. Assume $\mathbb{K}[G]$ is semisimple and consider the subspace

$$U \overset{\text{def}}{=} \mathbb{K}\text{-span}\left\{\sum_{g\in G} g\right\}.$$

This is a 1-dimensional subrepresentation of $\mathbb{K}[G]$.

By semisimplicity, there exists a complementary subrepresentation $V \subset \mathbb{K}[G]$ with $\mathbb{K}[G] = U \oplus V$.

View $\mathbb{K}$ as a G-representation with $g \cdot c = c$ for all $g \in G$ and $c \in \mathbb{K}$.

Then define $\phi : \mathbb{K}[G] \to \mathbb{K}$ to be linear map that sends $V \to 0$ and $\sum_{g\in G} g \mapsto 1_{\mathbb{K}}$.

Because U and V are subrepresentations, the map ϕ is a morphism of $\mathbb{K}[G]$-representations.

Thus $\phi(g) = \phi(g \cdot 1_G) = g \cdot \phi(1_G) = \phi(1_G) \in \mathbb{K}$ for all $g \in G$.

But this means that $1_{\mathbb{K}} = \phi(\sum_{g\in G} g) = \sum_{g\in G} \phi(g) = \sum_{g\in G} \phi(1_G) = |G|\phi(1_G)$.

Thus $|G|$ is invertible (and nonzero) in $\mathbb{K}$, so $\mathrm{char}(\mathbb{K})$ must not divide $|G|$. □

3 Characters of group representations

Continue to let G be a finite group.

If (ρ, V) is a G-representation with $\dim(V) < \infty$ then its *character* is the map $\chi_{(\rho,V)} : G \to \mathbb{K}$ with

$$\chi_{(\rho,V)}(g) = \mathrm{trace}(\rho(g)).$$

Since traces are invariant under change of basis, it follows that:

Fact. If $(\rho, V) \cong (\rho', V')$ as G-representations then $\chi_{(\rho,V)} = \chi_{(\rho',V')}$.

The *conjugacy classes* of G are the sets $\mathcal{K}_g \overset{\text{def}}{=} \{xgx^{-1} : x \in G\}$ for $g \in G$.

A *class function* of G is a map $G \to \mathbb{K}$ that is constant on all elements in each conjugacy class.

Equivalently, $f : G \to \mathbb{K}$ is a class function if and only if $f(xgx^{-1}) = f(g)$ for all $x, g \in G$.

Fact. The character of any finite-dimensional G-representation is a class function.

We say that the character $\chi_{(\rho,V)}$ is *irreducible* if (ρ, V) is an irreducible representation.

We mention some special properties of irreducible characters that hold when $\mathbb{K}[G]$ is semisimple.

Proposition. Assume $\mathrm{char}(\mathbb{K})$ does not divide $|G|$.

Then the irreducible characters of G are a basis for the vector space of class functions of G.

Proof. In this case $\mathbb{K}[G]$ is semisimple so the irreducible characters are a basis for $(\mathbb{K}[G]/[\mathbb{K}[G], \mathbb{K}[G]])^*$. This dual space can be identified with the vector space of linear maps $f : G \to \mathbb{K}$ that satisfy

$$f(XY) = f(YX) \qquad \text{for all } X, Y \in \mathbb{K}[G].$$

This is the same as the set of linear maps $f : G \to \mathbb{K}$ with

$$f(gh) = f(hg) \qquad \text{for all } g, h \in G$$

or equivalently with

$$f(xgx^{-1}) = f(g) \qquad \text{for all } x, g \in G.$$

Thus, we can identify $(\mathbb{K}[G]/[\mathbb{K}[G], \mathbb{K}[G]])^*$ with the vector space of class functions of G. □

Corollary. Assume $\mathrm{char}(\mathbb{K})$ does not divide $|G|$.

Then the number of isomorphism classes of irreducible G-representations is the same as the number of distinct irreducible characters of G, which is also the number of distinct conjugacy classes of G.

Corollary. If $\mathrm{char}(\mathbb{K}) = 0$ then finite-dimensional G-representations are isomorphic if and only if they have the same character.

A group G is *abelian* if $gh = hg$ for all $g, h \in G$.

This holds if and only if the group algebra $\mathbb{K}[G]$ is commutative, so the following is true:

Fact. If G is abelian then all irreducible G-representations are 1-dimensional.

Suppose $f : V \to W$ is a linear map between vector spaces.

Recall that V^* is the vector space of linear maps $\lambda : V \to \mathbb{K}$.

Define $f^* : W^* \to V^*$ to be the linear map with the formula $f^*(\lambda) = \lambda \circ f$.

If $f \in \mathrm{GL}(V)$ then $f^* \in \mathrm{GL}(V^*)$ since $(f \circ g)^* = g^* \circ f^*$.

Now suppose (ρ_V, V) is a G-representation. Define $\rho_{V^*} : G \to \mathrm{GL}(V^*)$ by the formula

$$\rho_{V^*}(g) = (\rho_V(g)^*)^{-1} = (\rho_V(g)^{-1})^* = \rho_V(g^{-1})^*.$$

Fact. If (ρ_V, V) is a representation then so is (ρ_{V^*}, V^*).

From this point on, we assume $\dim(V) < \infty$.

Fact. We have $\mathrm{trace}(f) = \mathrm{trace}(f^*)$ so $\chi_{(\rho^*,V^*)}(g) = \chi_{(\rho,V)}(g^{-1})$ for all $g \in G$.

Proposition. Suppose $\mathbb{K} = \mathbb{C}$. Then for all $g \in G$ it holds that

$$\overline{\chi_{(\rho_V,V)}(g)} = \chi_{(\rho_V,V)}(g^{-1}) = \chi_{(\rho_{V^*},V^*)}(g).$$

Therefore, we have $(\rho_V, V) \cong (\rho_{V^*}, V^*)$ if and only if $\chi_{(\rho_V,V)}$ takes all **real** values.

Proof. As G is a finite group, any $g \in G$ has $g^{|G|} = 1_G$, and so any eigenvalue of $\rho_V(g)$ is a root of unity.
The character value $\chi_{(\rho_V,V)}(g)$ is the sum of the eigenvalues of $\rho_V(g)$.
Hence this value is a sum of roots of unity in $\mathbb{K}$.

When $\mathbb{K} = \mathbb{C}$, the inverse of any root of unity is its complex conjugate.
As the eigenvalues of $\rho_V(g^{-1})$ are the inverses of the eigenvalues of $\rho_V(g)$, the result follows. □

Finally suppose (ρ_V, V) and (ρ_W, W) are G-representations.
Then $(\rho_{V\otimes W}, V \otimes W)$ is a G-representation if we define $\rho_{V\otimes W}(g)$ for $g \in G$ as the linear map sending

$$v \otimes w \mapsto \rho_V(g)(v) \otimes \rho_W(g)(w) \qquad \text{for } v \in V \text{ and } w \in W.$$

Fact. If $\dim(V) < \infty$ and $\dim(W) < \infty$ then $\chi_{(\rho_{V\otimes W},V\otimes W)} = \chi_{(\rho_V,V)}\chi_{(\rho_W,W)}$.