1 Review: Maschke's theorem and group characters

Let $\mathbb{K}$ be an algebraically closed field and let G be a group.

A *representation* of G is a representation (ρ, V) of the group algebra $\mathbb{K}[G]$.

This is the same data as a pair (ρ, V) such that $\rho : G \to \mathrm{GL}(V)$ is a group homomorphism.

Here V is required to be a $\mathbb{K}$-vector space, and we have $\rho(\mathbb{K}[G]) \subseteq \mathsf{End}(V)$ and $\rho(G) \subseteq \mathrm{GL}(V)$.

Now assume G is a finite group.

Theorem (*Maschke's theorem*)**.** The group algebra $\mathbb{K}[G]$ is *semisimple* (meaning that all irreducible G-representations are finite-dimensional and finite-dimensional G-representations are direct sums of irreducible representations) if and only if $\mathrm{char}(\mathbb{K})$ does not divide $|G|$.

Assume (ρ, V) is a finite dimensional G-representation.

The *character* of (ρ, V) is a linear map $\chi_{(\rho,V)} : \mathbb{K}[G] \to \mathbb{K}$ sending $g \mapsto \mathrm{trace}(\rho(g))$ for all $g \in G$.

Notice that $\dim(V) = \chi_{(\rho,V)}(1)$. Sometimes this is called the *degree* of the character.

Example. If (ρ, V) is a G-representation with $\dim(V) = 1$, then $\chi_{(\rho,V)} = \rho$.

We say that $\chi_{(\rho,V)}$ is *irreducible* if (ρ, V) is an irreducible representation.

Let $\mathrm{Irr}(G)$ denote the set of irreducible characters of G.

For any G-representations (ρ, V) and (ρ', V'), the following properties hold:

(1) If $(\rho, V) \cong (\rho', V')$ then $\chi_{(\rho,V)} = \chi_{(\rho',V')}$.

(2) The character $\chi_{(\rho,V)}$ is a *class function* on G, meaning that it is constant on conjugacy classes.

When $\mathbb{K}[G]$ is semisimple, some additional properties hold:

(3) $\mathrm{Irr}(G)$ is a basis for the $\mathbb{K}$-vector space of class functions $G \to \mathbb{K}$.

(4) If $\mathrm{char}(\mathbb{K}) = 0$, then $\chi_{(\rho,V)} = \chi_{(\rho',V')}$ if and only if $(\rho, V) \cong (\rho', V')$.

(5) $\sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 = |G|$.

Example. Suppose $\mathbb{K} = \mathbb{C}$ and G is a cyclic group of order $n \geq 1$ generated by an element x.

Let χ_m be the map $\mathbb{C}[G] \to \mathbb{C}$ with $x^j \mapsto \zeta^{mj}$ where $\zeta = e^{2\pi\sqrt{-1}/n}$.

Then $\mathrm{Irr}(G) = \{\chi_0, \chi_1, \chi_2, \dots, \chi_{n-1}\}$.

We can also form *duals* and *tensor products* of group representations.

When the representations are finite-dimensional, there are some related character formualas.

Remark. A G-representation is a left $\mathbb{K}[G]$-module. The algebra $\mathbb{K}[G]$ is often noncommutative.

Earlier, we emphasized that if A is a noncommutative algebra then the tensor product of two left A-modules is not a well-defined left A-module in general.

So how do we explain the existence of a tensor product for group representations?

The tensor product of two left A-modules always has the structure of a left $A \otimes A$-module.

So the tensor product of G-representations (ρ_V, V) and (ρ_W, W) is a representation of $\mathbb{K}[G] \otimes \mathbb{K}[G]$.

A special property of group algebras is that $\mathbb{K}[G] \otimes \mathbb{K}[G]$ has a subalgebra $\mathbb{K}\text{-span}\{g \otimes g : g \in G\} \cong \mathbb{K}[G]$.

By identifying $\mathbb{K}[G]$ with this subalgebra, any $\mathbb{K}[G] \otimes \mathbb{K}[G]$-representation becomes a $\mathbb{K}[G]$-representation.

This is how we define the G-representation $(\rho_V, V) \otimes (\rho_W, W)$.

2 More special properties of characters

For the rest of today, we assume that G is a finite group.

Suppose V and W are G-representations. Let $\mathrm{Hom}(W, V)$ denote the set of $\mathbb{K}$-linear maps $W \to V$.

The vector space $\mathrm{Hom}(W, V)$ is a left $\mathbb{K}[G] \otimes \mathbb{K}[G]$-module for the action

$$(g \otimes h) \cdot \varphi : w \mapsto g\varphi(h^{-1}w) \quad \text{for } g, h \in G.$$

Indeed, notice that if $\phi : W \to V$ is linear and $w \in W$ then for any $g_1, g_2, h_1, h_2 \in G$

$$\begin{aligned}\big((g_1g_2 \otimes h_1h_2) \cdot \varphi\big)(w) &= g_1g_2\varphi(h_2^{-1}h_1^{-1}w) \\ &= g_1\big(g_2 \otimes h_2 \cdot \varphi\big)(h_1^{-1}w) = \big((g_1 \otimes h_1)(g_2 \otimes h_2) \cdot \varphi\big)(w).\end{aligned}$$

Now assume that V and W are finite-dimensional.

Proposition. It holds that $V \otimes W^* \cong \mathrm{Hom}(W, V)$ as $\mathbb{K}[G] \otimes \mathbb{K}[G]$-modules.

Proof. Let $F : V \otimes W^* \to \mathrm{Hom}(W, V)$ be the linear map sending

$$v \otimes \varphi \mapsto (w \mapsto \varphi(w)v) \quad \text{for } v \in V \text{ and } \varphi \in W^*.$$

If $\{v_i\}$ is a basis for V, $\{w_j\}$ is basis for W, and $\{\delta_j\}$ is the dual basis for W^*, then F sends $v_i \otimes \delta_j$ to the linear map $W \to V$ whose matrix in the chosen bases has a 1 in position (i, j) and 0 everywhere else.

Any linear map $W \to V$ is a linear combination of such images $F(v_i \otimes \delta_j)$, so F is surjective.

Because

$$\dim(V \otimes W^*) = \dim(V)\dim(W^*) = \dim(V)\dim(W) = \dim(\mathrm{Hom}(W, V))$$

the map F is an isomorphism of $\mathbb{K}$-vector spaces.

For any $g, h \in G$, $v \in V$, $w \in W$, and $\varphi \in W^*$, we have

$$\big((g \otimes h) \cdot F(v \otimes \varphi)\big)(w) = g\varphi(h^{-1}w)v$$

which is the same as

$$F\big((g \otimes h) \cdot (v \otimes \varphi)\big)(w) = F((gv) \otimes (\varphi \circ h^{-1}))(w) = \varphi(h^{-1}w)(gv) = g\varphi(h^{-1}w)v.$$

Hence F is an isomorphism of $\mathbb{K}[G] \otimes \mathbb{K}[G]$-modules. □

By letting $g \in G$ act as $g \otimes g$, we can view $V \otimes W^* \cong \mathrm{Hom}(W, V)$ as isomorphic G-modules.

Let $\mathrm{Hom}_G(W, V) \subseteq \mathrm{Hom}(W, V)$ be the subspace of linear maps that commute with the action of G.

Proposition. The set $\mathrm{Hom}(W,V)^G$ of elements in $\mathrm{Hom}(W,V)$ fixed by all $g \in G$ is $\mathrm{Hom}_G(W,V)$.

Proof. If $\varphi \in \mathrm{Hom}_G(W,V)$ then for any $g \in G$ we have the following commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & V \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle g} \\ V & \xrightarrow{\varphi} & V \end{array}$$

and since the vertical map is invertible, it follows that $\varphi(w) = g(\varphi(g^{-1}w)) = (g \cdot \varphi)(w)$ for any $w \in W$. Thus, $\mathrm{Hom}_G(W,V) \subseteq \mathrm{Hom}(W,V)^G$.

Conversely, if $\varphi \in \mathrm{Hom}(W,V)^G$, then for any $g \in G$ and $w \in W$, we have

$$\varphi(gw) = (g \cdot \varphi)(gw) = g\varphi(g^{-1}gw) = g\varphi(w).$$

Thus, $\varphi \in \mathrm{Hom}_G(W,V)$ and $\mathrm{Hom}(W,V)^G \subseteq \mathrm{Hom}_G(W,V)$. □

Combining the preceding results lets us deduce that:

Corollary. There is an isomorphism $(V \otimes W^*)^G \cong \mathrm{Hom}_G(W,V)$ as G-modules.

Now assume $\mathbb{K} = \mathbb{C}$. For any maps $f_1, f_2 : G \to \mathbb{C}$, we define a positive-definite Hermitian form

$$(f_1, f_2) \overset{\text{def}}{=} \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)}.$$

Theorem. The set $\mathrm{Irr}(G)$ is an orthonormal basis for the class functions on G with respect to $(\cdot,\cdot)$.

In other words, for any $\chi, \psi \in \mathrm{Irr}(G)$ we have $(\chi, \psi) = \begin{cases} 1 & \text{if } \chi = \psi \\ 0 & \text{otherwise.} \end{cases}$

Proof. By Schur's Lemma, it suffices to prove that for any G-representations V and W, we have

$$(\chi_V, \chi_W) = \dim \mathrm{Hom}_G(W,V).$$

Let $\pi = \frac{1}{|G|}\sum_{g \in G} \in \mathbb{K}[G]$. Then

$$(\chi_V, \chi_W) = \frac{1}{|G|}\sum_{g \in G} \chi_V(g)\overline{\chi_W(g)} = \frac{1}{|G|}\sum_{g \in G} \chi_V(g)\chi_{W^*}(g) = \frac{1}{|G|}\sum_{g \in G} \chi_{V \otimes W^*}(g) = \chi_{V \otimes W^*}(\pi).$$

If X is any G-representation, then $X^G \overset{\text{def}}{=} \{x \in X : gx = x\}$ is a subrepresentation of G.

Notice that $g\pi = \frac{1}{|G|}\sum_{h \in G} gh = \frac{1}{|G|}\sum_{gh \in G} gh = \pi$ for any $g \in G$.

Therefore, we have $\pi x \in X^G$ for any $x \in X$ and $\pi : X \twoheadrightarrow X^G$ is a projection map.

Thus $\dim(X^G)$ is the character of X evaluated at π.

Applying this when $X = V \otimes W^*$ gives $\chi_{V \otimes W^*}(\pi) = \dim\left((V \otimes W^*)^G\right) = \dim(\mathrm{Hom}_G(W,V))$. □

For $g \in G$, let $Z_g = \{h \in G : hgh^{-1} = g\}$ be the *centralizer* of g.

Also let $\mathcal{K}_g = \{hgh^{-1} : h \in G\}$ be the conjugacy class of g.

Fact. By the *Orbit-Stabilizer Theorem* it holds that $|\mathcal{K}_g| = \frac{|G|}{|Z_g|}$.

Theorem. Let $g, h \in G$. Then

$$\sum_{\psi \in \mathrm{Irr}(G)} \psi(g)\overline{\psi(h)} = \begin{cases} |Z_g| & \mathcal{K}_g = \mathcal{K}_h \\ 0 & \mathcal{K}_g \neq \mathcal{K}_h. \end{cases}$$

Proof sketch. We want to describe this sum as the trace of a $\mathbb{C}$-endomorphism of $\mathbb{C}[G]$.

If we write V_ψ for an irreducible representation with character ψ, then we have

$$\begin{aligned} \sum_{\psi \in \mathrm{Irr}(G)} \psi(g)\overline{\psi(h)} &= \sum_{\psi \in \mathrm{Irr}(G)} \chi_{V_\psi}(g)\chi_{V_\psi^*}(h) \\ &= \sum_{\psi \in \mathrm{Irr}(G)} \chi_{V_\psi \otimes V_\psi^*}(g \otimes h) \\ &= \mathrm{trace}\left(\left(\bigoplus_{\psi \in \mathrm{Irr}(G)} \rho_{V_\psi \otimes V_\psi^*}\right)(g \otimes h)\right). \end{aligned}$$

We have an isomorphism $\bigoplus_{\psi \in \mathrm{Irr}(G)} V_\psi \otimes V_\psi^* \cong \oplus_{\psi \in \mathrm{Irr}(G)} \mathsf{End}(V_\psi) \cong \mathbb{C}[G]$ of $\mathbb{C}[G] \otimes \mathbb{C}[G]$ representations.

Under this isomorphism $g \otimes h$ acts on $\mathbb{C}[G]$ as the linear map sending $x \in G$ to gxh^{-1}.

Thus $\sum_{\psi \in \mathrm{Irr}(G)} \psi(g)\overline{\psi(h)}$ is the trace of $x \mapsto gxh^{-1}$ which is

$$|\{x \in G : x = gxh^{-1}\}| = |\{x \in G : g = xhx^{-1}\}| = \begin{cases} |Z_g| & \text{if } \mathcal{K}_g = \mathcal{K}_h \\ 0 & \text{if } \mathcal{K}_g \neq \mathcal{K}_h. \end{cases}$$

□

3 Unitary representations

A finite-dimensional representation (ρ, V) of a group G (over $\mathbb{C}$) is *unitary* if

$$(\rho(g)v, \rho(g)w) = (v, w) \qquad \text{for all } v, w \in V \text{ and } g \in G$$

for some positive-definite Hermitian form $(\cdot, \cdot) : V \times V \to \mathbb{C}$.

Proposition. If G is a finite group, then any finite dimensional G-representation is unitary.

Proof. Pick any basis $\{v_i\}$ for V. We define a positive-definite Hermitian form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ with

$$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Then the form $(v_i, v_j) \overset{\text{def}}{=} \sum_{g \in G} \langle gv_i, gv_j \rangle$ is positive-definite, Hermitian, and G-invariant. □

Proposition. Any finite-dimensional unitary representation of a (possibly infinite) group is semisimple.

Proof. Any irreducible representation is semisimple so assume V is reducible.

Choose an irreducible subrepresentation of $U \subsetneq V$. Write $(\cdot,\cdot)$ for the form that makes V unitary.

Let $U^\perp = \{v \in V : (v,u) = 0 \text{ for all } u \in U\}$.

Then $V = U \oplus U^\perp$ and $U^\perp$ is a subrepresentation since the relevant form is G-invariant.

The result now follows by induction on dimension. □

4 Matrix elements

Assume that G is a finite group and V is a finite-dimensional irreducible $\mathbb{C}[G]$-module.

Choose a G-invariant positive definite Hermitian form $(\cdot,\cdot)$ on V.

Let $\{v_i\}$ be an orthonormal basis on V with respect to $(\cdot,\cdot)$. Define

$$t^V_{ij}(g) = (gv_i, v_j) \qquad \text{and} \qquad \text{for } g \in G.$$

For each pair (i,j) with $1 \le i,j \le \dim V$, the map $t^V_{ij} : G \to \mathbb{C}$ is a called a *matrix element*.

Proposition. The rescaled matrix elements

$$\tfrac{1}{\sqrt{\dim V}} t^V_{ij} : G \to \mathbb{C}$$

give an orthonormal basis of the space of maps $G \to \mathbb{C}$ (as V ranges over all isomorphism classes of finite dimensional irreducible G-representations and i,j range over the indices of an orthonormal basis of V).

For a proof, see the textbook. Note that the number of distinct matrix elements is $\sum_V (\dim V)^2 = |G|$.

5 Character tables

Suppose G is a finite group. Choose representatives $1 = g_1, g_2, \cdots, g_r$ for the conjugacy classes in G.

Suppose $\mathbf{1} = \chi_1, \chi_2, \cdots, \chi_r$ are the distinct irreducible characters that make up $\mathrm{Irr}(G)$.

Here $\mathbf{1}$ denotes the irreducible character $G \to \{1\}$.

Then everything one wants to know about $\mathrm{Irr}(G)$ is encoded by the matrix

$\mathrm{Irr}(G)$	$1 = g_1$	g_2	$\cdots$	g_r
$\mathbf{1} = \chi_1$	1	1	$\cdots$	1
χ_2	$\chi_2(1)$	$\chi_2(g_2)$	$\cdots$	$\chi_2(g_r)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
χ_r	$\chi_r(1)$	$\chi_r(g_2)$	$\cdots$	$\chi_(g_r)$

which is called the *character table* of G.

Example. If $G = S_3$ then the character table of G is

$\mathrm{Irr}(S_3)$	1	$(1,2)$	$(1,2,3)$
$\chi_{(3)}$	1	1	1
$\chi_{(2,1)}$	2	0	-1
$\chi_{(1,1,1)}$	1	-1	1

Using the character table orthogonality relations from today, you can compute the sizes of all conjugacy classes in G. Then you can decompose arbitrary products of characters into irreducibles.