1 Review: orthogonality relations and character tables

Let G be a finite group and consider its representations over $\mathbb{C}$.

The vector space of functions $G \to \mathbb{C}$ has a positive definite Hermitian form

$$(f_1, f_2) \overset{\text{def}}{=} \tfrac{1}{|G|} \textstyle\sum_{g\in G} f_1(g)\overline{f_2(g)}.$$

Write $\mathrm{Irr}(G)$ for the set of (complex) irreducible characters of G.

Theorem. $\mathrm{Irr}(G)$ is an orthonormal basis for the space of class functions $G \to \mathbb{C}$ relative to $(\cdot,\cdot)$.

Corollary. If χ and ψ are any (complex) characters of G then $(\chi, \psi) \in \mathbb{N} = \{0, 1, 2, 3, \cdots\}$.

Proof. This holds as $\chi = \sum_{\phi\in\mathrm{Irr}(G)} a_\phi \phi$ and $\psi = \sum_{\phi\in\mathrm{Irr}(G)} b_\phi\phi$ where a_ϕ, b_ϕ are nonnegative integers. □

Corollary. If V and W are finite-dimensional representations of G over $\mathbb{C}$ then

$$(\chi_V, \chi_W) = \dim(\mathrm{Hom}_G(V, W)).$$

Proof. If V and W are irreducible then

$$\dim(\mathrm{Hom}_G(V, W)) = \begin{cases} 1 & V \cong W \\ 0 & V \not\cong W \end{cases}$$

by Schur's Lemma. If the irreducible decompositions of V and W are

$$V = \textstyle\bigoplus_{i=1}^r n_i V_i \qquad \text{and} \qquad W = \bigoplus_{j=1}^s m_j W_j$$

then $\dim \mathrm{Hom}_G(V, W) = \sum_{i,j} n_i m_j \mathrm{Hom}_G(V_i, W_j) = \sum_{i,j} n_i m_j(\chi_{V_i}, \chi_{W_j}) = (\chi_V, \chi_W)$. □

Theorem. Let $g, h \in G$ and define $Z_g = \{x \in G : xgx^{-1}\}$ and $\mathcal{K}_g = \{xgx^{-1} : x \in g\}$. Then

$$\sum_{\psi\in\mathrm{Irr}(G)} \psi(g)\overline{\psi(h)} = \begin{cases} |Z_g| & \mathcal{K}_g = \mathcal{K}_h \\ 0 & \mathcal{K}_g \neq \mathcal{K}_h. \end{cases}$$

Choose representatives $1 = g_1, g_2, \cdots, g_r$ for distinct conjugacy classes in G.

Suppose $\mathbf{1} = \chi_1, \cdots, \chi_r$ are the elements in $\mathrm{Irr}(G)$, with $\mathbf{1}$ the trivial character $G \to \{1\}$.

Then the *character table* of G is the matrix

$\mathrm{Irr}(G)$	g_1	g_2	$\cdots$	g_r
χ_1	1	1	1	1
χ_2	$\chi_2(1)$	$\chi_2(g_2)$	$\cdots$	$\chi_2(g_r)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
χ_r	$\chi_r(1)$	$\chi_r(g_2)$	$\cdots$	$\chi_(g_r)$

Suppose we are given the character table. The second theorem above let us compute $|Z_{g_i}|$ and $|\mathcal{K}_{g_i}|$.

Namely: we have $|Z_{g_1}| = |G| = \sum_{i=1}^r |\chi_i(1)|^2$ and $|Z_{g_i}| = \sum_{j=1}^r |\chi_j(g_i)|^2$ and then $|\mathcal{K}_{g_i}| = \frac{|G|}{|Z_{g_i}|}$.

This information and the first theorem lets us decompose any class function into irreducible characters. For if $f : G \to \mathbb{C}$ is a class function, then $f = \sum_{i=1}^{r} (f, \chi_i)\chi_i$ and the coefficients satisfy

$$(f, \chi_i) = \tfrac{1}{|G|} \sum_{g\in G} f(g)\chi_i(g) = \tfrac{1}{|G|} \sum_{j=1}^{r} f(g_j)\chi_i(g_j)|\mathcal{K}_{g_j}| = \sum_{j=1}^{r} \tfrac{\chi_i(g_j)}{|Z_{g_j}|} f(g_j).$$

Example. Suppose $G = S_3$ is the group of permutations of $\{1, 2, 3\}$. The character table of G is

$\mathrm{Irr}(S_3)$	1	$(1,2)$	$(1,2,3)$
$\chi_{(3)}$	1	1	1
$\chi_{(2,1)}$	2	0	-1
$\chi_{(1,1,1)}$	1	-1	1

The column indices of this table are permutations written in *cycle notation*.

The irreducible characters indexing the rows are labeled by *partitions* of the number 3.

Using the orthogonality relations above we compute that

$$|Z_1| = |S_3| = 1^2 + 2^2 + 1^2 = 6, \quad |Z_{(1,2)}| = 1^2 + 0^2 + 1^2 = 2, \quad \text{and} \quad |Z_{(1,2,3)}| = 1^2 + (-1)^2 + 1^2 = 3.$$

One can check that

$$\mathcal{K}_1 = \{1\}, \qquad \mathcal{K}_{(1,2)} = \{(1, 2), (1, 3), (2, 3)\}, \qquad \text{and} \qquad \mathcal{K}_{(1,2,3)} = \{(1, 2, 3), (1, 3, 2)\}.$$

It is clear that $\chi_{(3)}\psi = \psi$ for any class function ψ, while

$$(\chi_{(1,1,1)})^2 = \chi_{(3)} \qquad \text{and} \qquad \chi_{(1,1,1)}\chi_{(2,1)} = \chi_{(2,1)}.$$

But how does $\theta = (\chi_{(2,1)})^2$ decompose? This map's values on each conjugacy class are 4, 0, and 1 so

$$\big(\chi_{(3)}, \theta\big) = \tfrac{4}{6} + \tfrac{0}{2} + \tfrac{1}{3} = 1, \qquad \big(\chi_{(2,1)}, \theta\big) = \tfrac{8}{6} + \tfrac{0}{2} + \tfrac{-1}{3} = 1, \qquad \text{and} \qquad \big(\chi_{(1,1,1)}, \theta\big) = \tfrac{4}{6} + \tfrac{0}{2} + \tfrac{1}{3} = 1.$$

Therefore $(\chi_{(2,1)})^2 = \theta = \chi_{(3)} + \chi_{(2,1)} + \chi_{(1,1,1)}$.

2 Representations of product groups

Recall that if A and B are algebras over the same field, then the vector space $A \otimes B$ is an algebra.

If V is an A-representation and W is a B-representation, then $V \otimes W$ is an $A \otimes B$-representation.

The tensor product $V \otimes W$ is irreducible if and only if both V and W are irreducible.

Finally, every finite-dimensional irreducible representation of $A \otimes B$ arises as such a tensor product.

Definition. The *direct product* $G \times H$ of two groups G and H is the group

$$\{(g, h) : g \in G \text{ and } h \in H\} \qquad \text{and} \qquad \text{with product } (g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2).$$

Over any field $\mathbb{K}$, there is an isomorphism $\mathbb{K}[G \times H] \cong \mathbb{K}[G] \otimes \mathbb{K}[H]$ sending $(g, h) \in G \times H$ to $g \otimes h$.

Via this map, we can view any $\mathbb{K}[G] \otimes \mathbb{K}[H]$-representation as an algebra representation of $\mathbb{K}[G \times H]$.

In turn, the tensor product of a G-representation with an H-representation is a representation of $G \times H$.

Proposition. Let G and H be finite groups. Suppose $\{V_i\}_{i\in I}$ and $\{W_j\}_{j\in J}$ represent the distinct isomorphism classes of irreducible representations of G and H, respectively, over some field $\mathbb{K}$. Then

$$\{V_i \otimes W_j\}_{(i,j)\in I\times J}$$

represents the distinct isomorphism classes of irreducible representations of $G \times H$ over $\mathbb{K}$.

Proof. This is a special case of the statement for algebra because all irreducible representations of finite groups are finite-dimensional, as the associated group algebras are finite-dimensional. □

3 Restriction and induction

Suppose H is a subgroup of a group G. We consider representations here over an arbitrary field $\mathbb{K}$.

There is a simple way to convert any G-representation into a representation of H:

Definition. The *restriction* of a G-representation (ρ, V) is the H-representation

$$\mathrm{Res}_H^G(\rho, V) = (\rho|_H, V)$$

where $\rho|_H$ is the restriction of the homomorphism $\rho : G \to \mathrm{GL}(V)$ to H.
We write $\mathrm{Res}_H^G(\rho, V)$ as $\mathrm{Res}_H^G(V)$ when ρ is clear from context.

There is a less trivial way to turn an H-representation into a representation of G:

Definition. Given an H-representation (ρ, V) define $\mathrm{Ind}_H^G(V)$ to be the vector space

$$\mathrm{Ind}_H^G(V) = \Big\{ f : G \to V : f(hx) = \rho(h)f(x) \text{ for all } h \in H \text{ and } x \in G \Big\}$$

and let ρ^G be the map $G \to \mathrm{GL}(\mathrm{Ind}_H^G(V))$ with

$$\rho^G(g)f = \Big(\text{the map sending } x \mapsto f(xg)\Big).$$

Finally, define the *induced G-representation* of (ρ, V) to be the pair $\mathrm{Ind}_H^G(\rho, V) = (\rho^G, \mathrm{Ind}_H^G(V))$.
We write $\mathrm{Ind}_H^G(\rho, V)$ as $\mathrm{Ind}_H^G(V)$ when ρ is clear from context.

For the rest of today we let (ρ, V) denote some fixed H-representation.

Proposition. The pair $\mathrm{Ind}_H^G(\rho, V)$ is a G-representation.

Proof. Suppose $f \in \mathrm{Ind}_H^G(V)$ and $g, x \in G$ and $h \in H$. Write $g \cdot f$ in place of $\rho^G(g)f$. Then

$$(g \cdot f)(hx) = f(hxg) = f(h(xg)) = \rho(h)f(xg) = \rho(h)(g \cdot f)(x)$$

so $g \cdot f \in \mathrm{Ind}_H^G(V)$, and if $g_1, g_2 \in G$ then $g_1 \cdot (g_2 \cdot f) = (g_1 g_2) \cdot f$ since

$$(g_1 \cdot (g_2 \cdot f))(x) = f(xg_1g_2) = ((g_1g_2) \cdot f)(x).$$

□

A *left coset* of H in G is a set of the form $gH = \{gh : h \in H\}$ for some $g \in G$.

A *right coset* of H in G is a set of the form $Hg = \{hg : h \in H\}$ for some $g \in G$.

The group G is a disjoint union of the left cosets of H, and of the right cosets of H.

Let $X_L \subseteq G$ be a set of representatives for the distinct left cosets of H in G.

This means that each left coset has the form gH for exactly one $g \in X_L$.

Set $X_R = \{g^{-1} : g \in X_L\}$. Then X_R is a set of representatives for the distinct right cosets of H in G.

Define $|G : H| = |X_L| = |X_R|$. When G is finite we have $|G : H| = \frac{|G|}{|H|}$.

Example. Suppose $G = S_n$ is the group of permutations of the set $\{1, 2, 3, \dots, n\}$.

If $H \cong S_{n-1}$ is the subgroup of elements that fix n, then one possibility for X_L is

$$\{w \in S_n : w(1) < w(2) < \cdots < w(n-1)\} \qquad \text{which has size } n = \tfrac{n!}{(n-1)!} = \tfrac{|S_n|}{|S_{n-1}|}.$$

Proposition. Suppose $|G : H| < \infty$ and $\dim(V) < \infty$. Then $\dim(\mathrm{Ind}_H^G(V)) = |G : H| \dim(V)$.

Proof. Every map $X_R \to V$ uniquely extends to an element of $\mathrm{Ind}_H^G(V)$.

In fact, the set of maps $X_R \to V$ is isomorphic as a vector space to $\mathrm{Ind}_H^G(V)$.

But the dimension of the vector space of maps from any finite set S to V is $|S| \dim(V)$. $\square$

It is convenient to switch to module notation when working with induced representations.

Thus, we identify $\mathrm{Ind}_H^G(\rho, V)$ with $\mathrm{Ind}_H^G(V)$ and write $g \cdot f$ in place of $\rho^G(g)f$ for $g \in G$ and $f \in \mathrm{Ind}_H^G(V)$.

There is another way of constructing the G-module $\mathrm{Ind}_H^G(V)$ when G is finite.

This is convenient for some results, especially if one is comfortable with tensor products.

Specifically, by viewing $\mathbb{K}[G]$ as a $(\mathbb{K}[G], \mathbb{K}[H])$-bimodule and V as a left $\mathbb{K}[H]$-module we can form

$$\mathbb{K}[G] \otimes_{\mathbb{K}[H]} V$$

and this tensor product has the structure of a left $\mathbb{K}[G]$-module.

Concretely $\mathbb{K}[G] \otimes_{\mathbb{K}[H]} V$ is the quotient of the tensor product $\mathbb{K}[G] \otimes V$ by the subspace spanned by

$$gh \otimes v - g \otimes \rho(h)v \qquad \text{for all } g \in G,\ h \in H, \text{ and } v \in V.$$

The way G acts on this vector space is by multiplication on the left. If $\mathcal{B}$ is a basis for V then

$$\{g \otimes b\}_{(g,b) \in X_L \times \mathcal{B}} = \{g^{-1} \otimes b\}_{(g,b) \in X_R \times \mathcal{B}}$$

is a basis for $\mathbb{K}[G] \otimes_{\mathbb{K}[H]} V$.

Proposition. Assume G is a finite group. Then $\mathrm{Ind}_H^G(V) \cong \mathbb{K}[G] \otimes_{\mathbb{K}[H]} V$ as G-representations.

Proof. The map $\phi : \mathrm{Ind}_H^G(V) \to \mathbb{K}[G] \otimes_{\mathbb{K}[H]} V$ defined by $\phi(f) = \sum_{x \in G} x \otimes f(x^{-1})$ is an isomorphism.

The finiteness of G is needed for the definition of this map to make sense.

To check that ϕ is a morphism of G-representations, we compute for $g \in G$ and $f \in \mathrm{Ind}_H^G(V)$ that

$$g \cdot \phi(f) = \sum_{x \in G} gx \otimes f(x^{-1}) = \sum_{x \in G} x \otimes f(x^{-1}g) = \sum_{x \in G} x \otimes (g \cdot f)(x^{-1}) = \phi(g \cdot f).$$

To see that ϕ is bijective, observe that for any $f \in \mathrm{Ind}_H^G(V)$ we have

$$\begin{aligned}\phi(f) &= \sum_{x \in G} x \otimes f(x^{-1}) \\ &= \sum_{g \in X_L} \sum_{h \in H} gh \otimes f(h^{-1}g^{-1}) \\ &= \sum_{g \in X_L} \sum_{h \in H} g \otimes \rho(h) f(h^{-1}g^{-1}) = \sum_{g \in X_L} \sum_{h \in H} g \otimes f(g^{-1}) = |H| \sum_{g \in X_R} g^{-1} \otimes f(g).\end{aligned}$$

Since $f \in \mathrm{Ind}_H^G(V)$ has $f(g) = 0$ for all $g \in X_R$ only if $f = 0$, the map ϕ is injective.

Since any map $X_R \to V$ extends to an element $f \in \mathrm{Ind}_H^G(V)$, it follows that ϕ is also surjective. □

For our last result today, assume $|G| < \infty$ and $\dim(V) < \infty$.

Then the representation (ρ, V) has a character, which we denote by $\chi : G \to \mathbb{K}$.

As $\mathrm{Ind}_H^G(V)$ is also finite-dimensional in this case, it too has a character.

Proposition. Let $\mathrm{Ind}_H^G(\chi)$ be the character of $\mathrm{Ind}_H^G(V) = \mathrm{Ind}_H^G(\rho, V)$. Then for each $z \in G$ we have

$$\mathrm{Ind}_H^G(\chi)(z) = \sum_{\substack{g \in X_R \\ gzg^{-1} \in H}} \chi(gzg^{-1}) = \frac{1}{|H|} \sum_{\substack{g \in G \\ gzg^{-1} \in H}} \chi(gzg^{-1}),$$

where the second equality only holds when $\mathrm{char}(\mathbb{K})$ does not divide $|H|$.

Proof. Let B be a (finite) basis for V.

Consider G acting on $\mathbb{K}[G] \otimes_{\mathbb{K}[H]} V$ which has basis $\{g^{-1} \otimes v\}_{(g,v) \in X_R \times B}$.

We want to calculate the coefficient $a_{g,v} \in \mathbb{K}$ of $g^{-1} \otimes v$ in $z(g^{-1} \otimes v) = zg^{-1} \otimes v$ for $(g, v) \in X_R \times B$.

If $gzg^{-1} \notin H$ then $zg^{-1} \notin g^{-1}H$ so $a_{g,v} = 0$.

If $gzg^{-1} \in H$ then $zg^{-1} \otimes v = g^{-1}(gzg^{-1}) \otimes v = g \otimes \rho(gzg^{-1})v$ so $a_{g,v}$ is the coefficient of v in $\rho(gzg^{-1})v$.

Summing $a_{g,v}$ over $v \in B$ gives 0 if $gzg^{-1} \notin H$ and $\chi(gzg^{-1})$ if $gzg^{-1} \in H$.

Summing this over $g \in X_R$ gives the first desired formula.

If $\mathrm{char}(\mathbb{K})$ does not divide $|H|$, then $|H|$ is invertible in $\mathbb{K}$ and

$$\chi(gzg^{-1}) = \frac{1}{|H|} \sum_{h \in H} \chi(hgzg^{-1}h^{-1}) = \frac{1}{|H|} \sum_{x \in Hg} \chi(xzx^{-1})$$

whenever $g \in X_R$ has $gzg^{-1} \in H$. Summing this over $g \in X_R$ gives the second desired formula. □