# 1 Review: direct products, restriction, and induction

## 1.1 Product groups

The *direct product* $G \times H$ of two groups $G$ and $H$ is the group

$$\{(g,h) : g \in G \text{ and } h \in H\} \qquad \text{and} \qquad \text{with product } (g_1,h_1)(g_2,h_2) = (g_1g_2, h_1h_2).$$

Suppose $G$ and $H$ are finite groups and $\mathbb{K}$ is any field.

Choose any $G$-representation $V$ over $\mathbb{K}$ and any $H$-representation $W$ over $\mathbb{K}$.

Then the tensor product $V \otimes W$ is naturally a $G \times H$-representation via the formula

$$(g,h) : x \otimes y \mapsto \rho_V(g) \otimes \rho_W(h)y.$$

Moreover, the operation $(V,W) \to V \otimes W$ is a bijection

$$\{\text{irr. repns. of } G \text{ over } \mathbb{K}\} \times \{\text{irr. repns. of } H \text{ over } \mathbb{K}\} \xrightarrow{\sim} \{\text{irr. repns. of } G \times H \text{ over } \mathbb{K}\}.$$

## 1.2 Restriction and induction

Let $H$ be a subgroup of a finite group $G$. If $V$ is a $G$-representation over any field $\mathbb{K}$, then we write

$$\mathrm{Res}_H^G(V)$$

to denote $V$ viewed as an $H$-representation via restriction.

The character of $\mathrm{Res}_H^G(V)$ is $\chi_V|_H$, meaning the character of $V$ with domain restricted to $H$.

We refer to $\mathrm{Res}_H^G(V)$ as the *restriction* of $V$.

Now suppose $W$ is a representation of $H$ with corresponding map $\rho_W : H \to \mathrm{GL}(W)$.

Let $\mathrm{Ind}_H^G(W) = \left\{ f : G \to W : f(hx) = \rho_W(h)f(x) \text{ for all } h \in H \text{ and } x \in G \right\} \cong \mathbb{K}[G] \otimes_{\mathbb{K}[H]} W$.

Then $\mathrm{Ind}_H^G(W)$ is a $G$-representation (called the *induced representation*) for the action

$$g \cdot f : x \mapsto f(xg) \quad \text{for } g \in G \text{ and } f \in \mathrm{Ind}_H^G(W).$$

The induced representation has dimension $\frac{|G|}{|H|}\dim(W)$.

## 1.3 Induced characters

If $W$ is finite-dimensional with character $\chi_W$, then the character of $\mathrm{Ind}_H^G(W)$ has the formula

$$\mathrm{Ind}_H^G(\chi_W)(g) = \sum_{\substack{x \in X_L \\ x^{-1}gx \in H}} \chi_W(x^{-1}gx) \quad \text{for } g \in G,$$

where $X_L \subset G$ are any complete set of left coset representatives for $G/H = \{gH : g \in G\}$.

If $\mathrm{ch}(\mathbb{K})$ does not divide $|H|$ then this formula can be written as

$$\mathrm{Ind}_H^G(\chi_W)(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} \chi_W(x^{-1}gx).$$

Restriction and induction usually **do not** preserve irreducibility.

**Example.** Suppose $H = \{1_G\}$ is the trivial subgroup of $G$.

Let $\mathbf{1}_H : 1_G \mapsto 1_{\mathbb{K}}$ be the trivial character of $H$, which is also a 1-dimensional representation.

Then we can take $X_L = G$ and since $x^{-1}gx = 1_G$ if and only if $g = 1_G$ we get

$$\mathrm{Ind}_H^G(\mathbf{1}_H)(g) = \sum_{\substack{x\in G \\ x^{-1}gx\in H}} \mathbf{1}_H(x^{-1}gx) = \begin{cases} |G| & \text{if } g = 1_G \\ 0 & \text{otherwise.} \end{cases}$$

This is the character of the *regular representation* of $G$, so $\mathrm{Ind}_H^G(\mathbf{1}_H) \cong \mathbb{K}[G]$ as $G$-representations.

We could also see this by observing that $\mathrm{Ind}_H^G(\mathbf{1}_H) \cong \mathbb{K}[G] \otimes_{\mathbb{K}[H]} \mathbb{K} \cong \mathbb{K}[G] \otimes_{\mathbb{K}} \mathbb{K} \cong \mathbb{K}[G]$.

We mention a useful property, whose proof is a homework exercise.

**Proposition.** If $N \subset H \subset G$ are subgroups then obviously

$$\mathrm{Res}_N^G = \mathrm{Res}_N^H \mathrm{Res}_H^G,$$

and if $V$ is an $N$-representation then we likewise have $\mathrm{Ind}_N^G(V) \cong \mathrm{Ind}_H^G \mathrm{Ind}_N^H(V)$.

## 2 Frobenius reciprocity

Continue to let $G$ be a finite group with a subgroup $H$.

Suppose $V$ is a $G$-representation and $W$ is an $H$-representation, both over the same field $\mathbb{K}$.

Recall that if $V'$ is another $G$-representation then we write

$$\mathrm{Hom}_G(V, V') = \{\text{linear maps } \phi : V \to V' \text{ such that } \phi \circ \rho_V(g) = \rho_{V'}(g) \circ \phi \text{ for all } g \in G\}.$$

**Theorem** (Frobenius reciprocity)**.** Then $\mathrm{Hom}_G(V, \mathrm{Ind}_H^G(W)) \cong \mathrm{Hom}_H(\mathrm{Res}_H^G(V), W)$ as vector spaces.

One can make a more precise claim that there is a *natural isomorphism* between these two vector spaces such that the following diagrams commute for all morphisms $V \to V'$ and $W \to W'$:

$$\begin{array}{ccc} \mathrm{Hom}_G(V, \mathrm{Ind}_H^G(W)) & \xrightarrow{\sim} & \mathrm{Hom}_H(\mathrm{Res}_H^G(V), W) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_G(V, \mathrm{Ind}_H^G(W')) & \xrightarrow{\sim} & \mathrm{Hom}_H(\mathrm{Res}_H^G(V), W') \end{array} \qquad \begin{array}{ccc} \mathrm{Hom}_G(V, \mathrm{Ind}_H^G(W)) & \xrightarrow{\sim} & \mathrm{Hom}_H(\mathrm{Res}_H^G(V), W) \\ \uparrow & & \uparrow \\ \mathrm{Hom}_G(V', \mathrm{Ind}_H^G(W)) & \xrightarrow{\sim} & \mathrm{Hom}_H(\mathrm{Res}_H^G(V'), W) \end{array}$$

We mention one corollary of the theorem before explaining its proof.

**Corollary.** Suppose $K = \mathbb{C}$ and that $V$ and $W$ are both finite-dimensional.

Write $\chi_V$ and $\chi_W$ for the characters of $V$ and $W$. Then $(\chi_V, \mathrm{Ind}_H^G(\chi_W)) = (\mathrm{Res}_H^G(\chi_V), \chi_W)$.

*Proof.* From results last time, this just says that $\dim \mathrm{Hom}_G(V, \mathrm{Ind}_H^G(W)) = \dim \mathrm{Hom}_H(\mathrm{Res}_H^G(V), W)$.

This assertion follows from Frobenius reciprocity. □

# 3 Proof of Frobenius reciprocity

To reduce the number of parentheses we write $\mathrm{Res}_H^G V$ and $\mathrm{Ind}_H^G W$ instead of $\mathrm{Res}_H^G(V)$ and $\mathrm{Ind}_H^G(W)$ We view elements of $\mathrm{Ind}_H^G W$ as maps $G \to W$ rather than as elements of a tensor product.

Define $\Phi : \mathrm{Hom}_G(V, \mathrm{Ind}_H^G W) \to \mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$ by the formula

$$\Phi(\alpha) = \Big(v \mapsto \alpha(v)(1_G) \in W\Big).$$

Then define $\Psi : \mathrm{Hom}_H(\mathrm{Res}_H^G V, W) \to \mathrm{Hom}_G(V, \mathrm{Ind}_H^G W)$ by the formula

$$\Psi(\beta) = \Big(v \mapsto \Big(\text{the map } G \to W \text{ sending } x \mapsto \beta\Big(\underbrace{\rho_V(x)v}_{\in V}\Big) \in W\Big)\Big).$$

We deduce the theorem by checking the following claims.

**Claim.** If $\alpha \in \mathrm{Hom}_G(V, \mathrm{Ind}_H^G W)$ then $\Phi(\alpha)$ is a morphism of $H$-representations.

*Proof.* Observe for $\alpha \in \mathrm{Hom}_G(V, \mathrm{Ind}_H^G W)$ and $h \in H$ and $v \in V$ that

$$\Phi(\alpha)\Big(\rho_V(h)v\Big) = \alpha\Big(\rho_V(h)v\Big)(1_G) = \Big(h \cdot \alpha(v)\Big)(1_G) = \alpha(v)\Big(h \cdot 1_G\Big) = \rho_W(h) \cdot \alpha(v)(1_G) = \rho_W(h) \cdot \Phi(\alpha)(v).$$

□

**Claim.** If $\beta \in \mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$ then $\Psi(\beta)$ is a linear map $V \to \mathrm{Ind}_H^G W$.

*Proof.* We have defined $\Psi(\beta)$ to assign a vector $v \in V$ to a function $G \to W$.
We just need to check that that this function has the property required to be in $\mathrm{Ind}_H^G W$.
For this, observe for $v \in V$ and $h \in H$ and $x \in G$ that

$$\Big(\Psi(\beta)(v)\Big)(hx) = \beta\Big(\rho_V(hx)v\Big) = \beta\Big(\rho_V(h)\rho_V(x)v\Big) = \rho_W(h) \cdot \beta\Big(\rho_V(x)v\Big) = \rho_W(h) \cdot \Big(\Psi(\beta)(v)\Big)(x).$$

□

**Claim.** If $\beta \in \mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$ then $\Psi(\beta)$ is a morphism of $G$-representations.

*Proof.* It suffices to check that when $g, x \in G$ and $v \in V$ we have

$$\Psi(\beta)\Big(\rho_V(g)v\Big) : x \mapsto \beta\Big(\rho_V(x)\rho_V(g)x\Big) = \beta\Big(\rho_V(xg)v\Big) = \Big(\Psi(\beta)(v)\Big)(xg) = \Big(g \cdot \Psi(\beta)(v)\Big)(x)$$

as this shows that $\Psi(\beta)\Big(\rho_V(g)v\Big) = \Big(g \cdot \Psi(\beta)\Big)(v)$. □

**Claim.** The composition $\Phi \circ \Psi$ is the identity map on $\mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$.

*Proof.* Notice for $\beta \in \mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$ and $v \in V$ that we have

$$\Big(\Phi \circ \Psi(\beta)\Big)(v) = \Big(\Psi(\beta)(v)\Big)(1_G) = \beta(1_G v) = \beta(v).$$

Thus $\Phi \circ \Psi(\beta) = \beta$. □

**Claim.** The composition $\Psi \circ \Phi$ is the identity map on $\mathrm{Hom}_H(V, \mathrm{Ind}_H^G W)$.

*Proof.* Notice for $\alpha \in \mathrm{Hom}_H(V, \mathrm{Ind}_H^G W)$ and $v \in V$ and $x \in G$ that we have

$$\Big(\Psi \circ \Phi(\alpha)\Big)(v) : x \mapsto \Phi(\alpha)\Big(\rho_V(x)v\Big) = \Big(\alpha(\rho_V(x)v)\Big)(1_G) = \Big(x \cdot \alpha(v)\Big)(1_G) = \alpha(v)(x).$$

Thus $\Psi \circ \Phi(\alpha) = \alpha$. □

These claims show that $\Phi$ and $\Psi$ are inverse isomorphisms between $\mathrm{Hom}_G(V, \mathrm{Ind}_H^G W)$ and $\mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$.

Thus, we have established the Frobenius reciprocity theorem.

**Example.** Suppose $\mathbb{K} = \mathbb{C}$ and $\mathbf{1}_H$ is the trivial representation of the trivial subgroup $H = \{1_G\}$ of $G$.

Then for all irreducible characters $\chi \in \mathrm{Irr}(G)$ we have

$$(\mathrm{Ind}_H^G(\mathbf{1}_H), \chi) = (\mathbf{1}_H, \mathrm{Res}_H^G(\chi)) = \tfrac{1}{|H|} \sum_{h \in H} \mathbf{1}_H(h)\overline{\chi(h)} = \tfrac{1}{1} \cdot \mathbf{1}_H(1_G) \cdot \overline{\chi(1_G)} = \chi(1_G).$$

The value of $\chi(1_G)$ is the (positive integer) dimension of any representation with character $\chi$.

We have already seen that $\mathrm{Ind}_H^G(\mathbf{1}_H)$ is the character of the regular representation of $G$.

Thus, each irrep of $G$ appears in the regular representation with multiplicity equal to its dimension.

This is something we already knew from the theory of semisimple algebras $A \cong \bigoplus_{\text{irreducible } V} \mathsf{End}(V)$.

# 4 Representations of symmetric groups

Let $n$ be a positive integer and let $[n] = \{1, 2, \dots, n\}$.

Define $S_n$ to be the group of bijections $[n] \to [n]$. We call this the *symmetric group* on $n$ letters.

The elements of $S_n$ are often called *permutations* of the set $[n]$.

A *partition* of $n$ is a sequence of integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ where

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0 \quad \text{and} \quad \lambda_1 + \lambda_2 + \cdots + \lambda_k = n.$$

The number $k$ is called the *length* of $\lambda$ and is denoted $\ell(\lambda) = k$.

This length can be any integer between 1 and $n$.

When $k = 1$, when we must have $\lambda = (n)$ and when $k = n$ when we must have $\lambda = (1, 1, \dots, 1)$.

The positive numbers $\lambda_i$ are called the *parts* of $\lambda$.

We write $\lambda \vdash n$ to denote that $\lambda$ is a partition of $n$.

It is convenient to set $\lambda_i = 0$ if $i > \ell(\lambda)$ and to not distinguish between the finite and infinite sequences

$$(\lambda_1, \lambda_{,2}, \dots, \lambda_k) \quad \text{and} \quad (\lambda_1, \lambda_{,2}, \dots, \lambda_k, 0, 0, 0, \dots).$$

The *(Young) diagram* of a partition $\lambda \vdash n$ is the set of positions

$$\mathsf{D}_\lambda = \{(i,j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} : 1 \leq j \leq \lambda_i\}.$$

We view $\mathsf{D}_\lambda$ as a set of boxes oriented as in an $n \times n$ matrix.

Then we can refer to the rows and columns in $\mathsf{D}_\lambda$ like in a matrix.

**Example.** If $\lambda = (3, 3, 2)$, then $\mathsf{D}_\lambda = \begin{array}{|c|c|c|}\hline \phantom{1} & \phantom{1} & \phantom{1} \\ \hline \phantom{1} & \phantom{1} & \phantom{1} \\ \hline \phantom{1} & \phantom{1} \\ \cline{1-2} \end{array} = \left\{ \begin{array}{lll} (1,1), & (1,2), & (1,3), \\ (2,1), & (2,2), & (2,3), \\ (3,1), & (3,2) & \end{array} \right\}.$

A *tableau* of shape $\lambda$ is a map $T : \mathsf{D}_\lambda \to \mathbb{Z}$, which we view as a filling of the boxes in $\mathsf{D}_\lambda$ by integer values.

A *standard tableau* of shape $\lambda \vdash n$ is a bijection $T : \mathsf{D}_\lambda \mapsto [n]$ such that

- all rows of $T$ are increasing left-to-right, and
- all columns of $T$ are increasing top-to-bottom.

Formally, this means $T(i, j) < T(i, j+1)$ and $T(i, j) < T(i+1, j)$ when the relevant positions are in $\mathsf{D}_\lambda$.

**Example.** If $\lambda = (2, 2) \vdash n = 4$, then $T$ could be either

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

but there are no other standard tableaux of this shape.

Let $T_\lambda$ be the particular tableau of shape $\lambda$ whose entry in box $(i, j) \in \mathsf{D}_\lambda$ is

$$\lambda_1 + \lambda_2 + \cdots + \lambda_{i-1} + j.$$

For example if $\lambda = (4, 2, 1) \vdash n = 7$ then

$$T_\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array}.$$

Now, given $\lambda \vdash n$, let $P_\lambda$ be the subset of $\sigma \in S_n$ such that if $\sigma(i) = j$ then $i$ and $j$ are in same row of $T_\lambda$.

Also let $Q_\lambda$ be the subset of $\sigma \in S_n$ such that if $\sigma(i) = j$ then $i$ and $j$ are in same column of $T_\lambda$.

**Fact.** Both $P_\lambda$ and $Q_\lambda$ are subgroups of $S_n$ and $P_\lambda \cap Q_\lambda = \{1\}$.

There is a unique group homomorphism $\mathrm{sgn} : S_n \to \{\pm 1\}$ that assigns each transposition $(i\ j) \mapsto -1$.

This is called the *sign representation*. One can show that the value of $\mathrm{sgn}(\sigma)$ is the determinant of the $n \times n$ *permutation matrix* that has 1 in each position $(\sigma(j), j)$ for $j \in [n]$ and 0 in all other positions.

Finally, define the *Young projectors* in $\mathbb{Z}[S_n]$ to be

$$a_\lambda = \textstyle\sum_{g \in P_\lambda} g \in \mathbb{Z}[S_n] \quad \text{and} \quad b_\lambda = \textstyle\sum_{g \in Q_\lambda} \mathrm{sgn}(g) g \in \mathbb{Z}[S_n].$$

Then let $\boxed{c_\lambda = a_\lambda b_\lambda}$ and $\boxed{V_\lambda = \mathbb{C}[S_n] c_\lambda} = \mathbb{C}\text{-span}\{\sigma c_\lambda : \sigma \in S_n\} \subseteq \mathbb{C}[S_n]$.

The irreducible representations of $S_n$ over $\mathbb{C}$ are described by the following theorem.

**Theorem.** The subspace $V_\lambda \subset \mathbb{C}[S_n]$ is an irreducible $S_n$-module, called a *Specht module*.

Conversely, each irreducible complex representation of $S_n$ is isomorphic to $V_\lambda$ for a unique partition $\lambda \vdash n$.

We will outline a proof of this theorem in the next lecture.

**Corollary.** All irreducible complex representations of $S_n$ are realizable over $\mathbb{Q}$, meaning that there exists a basis such that the matrices corresponding to all group elements have all entries in $\mathbb{Q}$.

**Remark.** What happens if we work over an algebraically closed field $\mathbb{K}$ with $\mathrm{ch}(\mathbb{K}) = p > 0$?

(1) You will still have a construction of Specht modules $V_\lambda$, but they are not always irreducible.

(2) There is an associated symmetric bilinear from $V_\lambda \times V_\lambda \to \mathbb{K}$ and the quotients

$$D_\lambda = V_\lambda/(\text{radical of the form})$$

are each irreducible $S_n$-modules or zero.

(3) However, it is hard to detect when $D_\lambda \neq 0$ and to compute a formula for $\dim(D_\lambda)$.

(4) Finally, there is now a distinction between irreducible and indecomposable $S_n$-representations.