1 Review: Frobenius recipocity

Let $H \subseteq G$ be finite groups.

Suppose V is a G-representation and W is an H-representation over the same field $\mathbb{K}$.

Then $\dim \operatorname{Hom}_G\Big(V, \operatorname{Ind}_H^G(W)\Big) = \dim \operatorname{Hom}_H\Big(\operatorname{Res}_H^G(V), W\Big)$.

This statement about dimensions can be rephrased as the existence of certain natural isomorphism between the relevant vector spaces of morphisms.

Corollary. If $\mathbb{K} = \mathbb{C}$ and V and W are finite-dimensional with characters χ_V and χ_W then

$$\Big(\chi_V, \operatorname{Ind}_H^G(\chi_W)\Big)_G = \Big(\operatorname{Res}_H^G(\chi_V), \chi_W\Big)_H$$

where $(f,g)_X = \frac{1}{|X|}\sum_{x\in X} f(x)\overline{g(x)}$ for $X = G$ or $X = H$.

2 Classifying the irreducible representations of S_n

Let n be a positive integer. Let S_n be the *symmetric group* of permutations of $[n] = \{1, 2, \dots, n\}$.

Last time we gave a construction of all irreducible representations of the symmetric group S_n over $\mathbb{C}$.

Our goal today is to prove this result.

Recall that a *partition* $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ is a weakly decreasing sequence of positive integers.

Set $\ell(\lambda) = k$ and write $\lambda \vdash n$ if $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$.

The *diagram* of a partition λ is the set

$$\mathsf{D}_\lambda = \{(i,j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} : 1 \leq j \leq \lambda_i\}$$

which we view as a subset of positions in a matrix, often drawn as boxes.

This convention lets us refer to rows, columns, and diagonals in the diagram.

For example, if $\lambda = (4,1,1)$ then $\mathsf{D}_\lambda = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & \multicolumn{3}{c}{} \\ \cline{1-1} & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array} = \{(1,1), (1,2), (1,3), (1,4), (2,1), (3,1)\}$.

A *tableau* of shape λ is a map $T : \mathsf{D}_\lambda \to \mathbb{Z}$ which we think of as a partially filled-in matrix.

A tableau is *standard* if the entries are the numbers $1, 2, 3, \cdots, n$ with no repetitions, such that all rows and columns are increasing from left-to-right and top-to-bottom.

Some examples of standard tableaux of shape $\lambda = (4,2,1) \vdash n = 7$ are

$$T_\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 7 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 7 \\ \hline 2 & 5 & \multicolumn{2}{c}{} \\ \cline{1-2} 3 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 & \multicolumn{2}{c}{} \\ \cline{1-2} 6 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}.$$

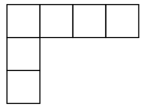
Define T_λ to be the **unique** standard tableau of shape λ whose rows all contain **consecutive** integers.

For $\lambda \vdash n$ the *row and column stabilizer subgroups* are

$$P_\lambda = \{\sigma \in S_n : \sigma(i) = j \text{ if and only if } i \text{ and } j \text{ are in the same row of } T_\lambda\},$$
$$Q_\lambda = \{\sigma \in S_n : \sigma(i) = j \text{ if and only if } i \text{ and } j \text{ are in the same column of } T_\lambda\}.$$

Fact. Let λ'_i be the number of cells in column i of D_λ. Set $k = \ell(\lambda)$ and $p = \lambda_1$. Then

$$P_\lambda \cong S_{\lambda_1} \times S_{\lambda_2} \times \cdots S_{\lambda_k} \quad \text{and} \quad Q_\lambda \cong S_{\lambda'_1} \times S_{\lambda'_2} \times \cdots S_{\lambda'_p}.$$

For example, if $\lambda = (4, 1, 1) =$  then $k = 3$ and $p = 4$ and $(\lambda'_1, \lambda'_2, \lambda'_3, \lambda'_4) = (3, 1, 1, 1)$.

The *sign representation* is the unique group homomorphism

$$\mathrm{sgn} : S_n \to \{\pm 1\}$$

with $\boxed{\mathrm{sgn}((i\ j)) = -1}$ for $i \neq j$ and $\boxed{\mathrm{sgn}((i_1\ i_2\ i_3\ \cdots i_m)) = (-1)^{m-1}}$ if $i_1, i_2, \dots, i_m \in [n]$ are all distinct.

Given $\lambda \vdash n$ define $\boxed{c_\lambda = a_\lambda b_\lambda}$ where

$$a_\lambda = \textstyle\sum_{g \in P_\lambda} g \in \mathbb{Z}[S_n] \quad \text{and} \quad b_\lambda = \textstyle\sum_{g \in Q_\lambda} \mathrm{sgn}(g) g \in \mathbb{Z}[S_n].$$

(These definitions differ from the textbook by a constant factor.)

Then let $\boxed{V_\lambda = \mathbb{C}[S_n] c_\lambda} = \mathbb{C}\text{-span}\{\sigma c_\lambda : \sigma \in S_n\}$. We call this left S_n-module a *Specht module*.

We spend the rest of this lecture proving the following theorem:

Theorem. Each V_λ for $\lambda \vdash n$ is an irreducible S_n-module.

Conversely, each irreducible complex representation of S_n is isomorphic to V_λ for a unique $\lambda \vdash n$.

Example. If $\lambda = (n)$ then $P_\lambda = S_n$ and $Q_\lambda = \{1\}$ so $c_\lambda = \sum_{g \in S_n} g$.

In this case $\sigma c_\lambda = c_\lambda$ for all $\sigma \in S_n$ so $V_{(n)} \cong \mathbf{1}$ is the 1-dimensional trivial representation of S_n.

Example. If $\lambda = (1, 1, \dots, 1)$ then $P_\lambda = \{1\}$ and $Q_\lambda = S_n$ so $c_\lambda = \sum_{g \in S_n} \mathrm{sgn}(g) g$.

In this case $\sigma c_\lambda = \mathrm{sgn}(\sigma) c_\lambda$ for all $\sigma \in S_n$ so $V_{(1,1,\dots,1)}$ is the 1-dimensional *sign representation* of S_n.

Remark. Suppose $p_1, p_2 \in P_\lambda$ and $q_1, q_2 \in Q_\lambda$ are such that $p_1 q_1 = p_2 q_2$. Then

$$p_2^{-1} p_1 = q_2 q_1^{-1} \in P_\lambda \cap Q_\lambda.$$

Since we have Since $P_\lambda \cap Q_\lambda = \{1\}$ it must hold that $p_1 = p_2$ and $q_1 = q_2$.

Thus any $g \in P_\lambda Q_\lambda$ has a unique factorization $g = pq$ for some $p \in P_\lambda$ and $q \in Q_\lambda$.

We will prove the theorem through a sequence of lemmas.

Lemma. Suppose $g \in S_n$. Then the following properties hold:

(1) If $g \in P_\lambda Q_\lambda$ then $g = pq$ for unique elements $p \in P_\lambda$ and $q \in Q_\lambda$ and $a_\lambda g b_\lambda = \mathrm{sgn}(q) c_\lambda$.

(2) If $g \notin P_\lambda Q_\lambda$ then $a_\lambda g\, b_\lambda = 0$.

Proof. Part (1) holds by the remark and the observation that $a_\lambda g b_\lambda = a_\lambda p q b_\lambda = a_\lambda \mathrm{sgn}(q) b_\lambda = \mathrm{sgn}(q) c_\lambda$.

Proving part (2) is harder. We start with the following observation:

- If some $t = (i\ j) \in S_n$ with $1 \leq i < j \leq n$ has $t \in P_\lambda$ and $g^{-1}tg \in Q_\lambda$ then $a_\lambda g b_\lambda = 0$ because

$$a_\lambda g b_\lambda = a_\lambda t g b_\lambda = a_\lambda g g^{-1} t g b_\lambda = -a_\lambda g b_\lambda.$$

Thus it suffices to show that if there are no such transpositions t then $g \in P_\lambda Q_\lambda$.

Let $T = T_\lambda$ and defined $T' = gT$ to be the tableau formed by applying g to each entry of T.

For example, if $\lambda = (4, 2, 1)$ and $g = (1\ 3\ 4)(6\ 7) \in S_7$ then

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array} \quad \text{then} \quad T' = \begin{array}{|c|c|c|c|}\hline 3 & 2 & 4 & 1 \\ \hline 5 & 7 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}.$$

A transposition $t = (i, j)$ has the properties noted above if and only if i and j are in the same row of T and in same column of T'. We see for the example given that $t = (5, 6)$ has these properties.

Suppose that no such transpositions $i, j \in [n]$ exist.

Then any two elements in the first row of T belong to different columns of T'.

Hence there exists $p_1 \in P_\lambda$ and $q_1' \in gQ_\lambda g^{-1}$ such that $p_1 T$ and $q_1' T'$ have the same first row.

By repeating this argument for the second, third, and remaining rows successively, we conclude by induction on the number of rows that there exists $p \in P_\lambda$ and $q' \in gQ_\lambda g^{-1}$ such that $pT = q'T'$.

But this means that $pT = q'gT = gqT$ for $q = g^{-1}q'g \in Q_\lambda$.

Finally observe that we can only have $g_1 T = g_2 T$ if $g_1 = g_2$ in S_n.

Hence we must have $p = gq$, which implies that $g = pq^{-1} \in P_\lambda Q_\lambda$ as we needed to show. □

The *lexicographic order* on partitions is the total order with $\mu < \lambda$ if and only if there exists j with

$$\mu_j < \lambda_j \quad \text{and} \quad \mu_i = \lambda_i \text{ for all } 1 \leq i < j$$

where we set $\mu_i = 0$ for $i > \ell(\mu)$. On partitions of $n = 4$ this order is

$$(1, 1, 1, 1) < (2, 1, 1) < (2, 2) < (3, 1) < (4).$$

Lemma. Assume $\lambda, \mu \vdash n$ and $\lambda > \mu$ in lexicographic order. Then $a_\lambda \mathbb{C}[S_n] b_\mu = 0$.

Proof. Fix an element $g \in S_n$.

It suffices to show that there exists a transposition $t = (i\ j) \in S_n$ with $t \in P_\lambda$ and $g^{-1}tg \in Q_\mu$ as then

$$a_\lambda g b_\mu = a_\lambda t g b_\mu = a_\lambda g g^{-1} t g b_\mu = -a_\lambda g b_\mu.$$

To this end let $T = T_\lambda$ and $T' = gT_\mu$.

Claim. There are numbers $a < b$ appearing in the same row of T and the same column of T'.

Proof of this claim. Let j be the first index with $\mu_j < \lambda_j$, so that $\mu_i = \lambda_i$ for all $1 \leq i < j$.
If $j = 1$ then our claim must hold by the pigeonhole principle.

Suppose $j > 1$ and any two elements of the first row of T are in different columns of T'.
Then we can find $p \in P_\lambda$ and $q' \in gQ_\mu g^{-1}$ such that pT and $q'T'$ have same first row.

Repeat this argument for the second, third, and remaining rows successively.
We conclude by induction on j that the claim holds. ■

The transposition $t = (a\ b)$ with $a < b$ as in the claim has the desired properties so $a_\lambda \mathbb{C}[S_n] b_\mu = 0$. □

Lemma. It holds that $c_\lambda^2 = \dfrac{n!}{\dim V_\lambda} c_\lambda$.

Proof. Since $a_\lambda g b_\lambda$ is either $\pm a_\lambda b_\lambda = \pm c_\lambda$ or 0 for each $g \in S_n$, we have

$$c_\lambda^2 = a_\lambda (b_\lambda a_\lambda) b_\lambda \in \mathbb{Z}\text{-span}\{a_\lambda g b_\lambda : g \in Q_\lambda P_\lambda\} \subseteq \mathbb{Z}\text{-span}\{c_\lambda\}.$$

Thus $c_\lambda^2 = K c_\lambda$ for some scalar $K \in \mathbb{Z}$.

Consider the linear map $L : \mathbb{C}[S_n] \to \mathbb{C}[S_n]$ given by $L(x) = xc_\lambda$.
The coefficient of the identity element in

$$c_\lambda = \left(\textstyle\sum_{p \in P_\lambda} p\right) \left(\textstyle\sum_{q \in Q_\lambda} \mathrm{sgn}(q) q\right)$$

is 1 since the only way to express $1 = pq$ with $p \in P_\lambda$ and $q \in Q_\lambda$ is by taking $p = q = 1$.
Thus the coefficient of g in $L(g) = gc_\lambda$ is also 1 for each $g \in G$.
This means the trace of $L : \mathbb{C}[S_n] \to \mathbb{C}[S_n]$ is $\mathrm{trace}(L) = |S_n| = n!$.

This trace is the sum of the eigenvalues of L, which would all be zero if $c_\lambda^2 = 0$, so we must have $K \neq 0$.
Thus $\frac{1}{K} L : x \mapsto \frac{1}{K} x c_\lambda$ is an idempotent linear map $\mathbb{C}[S_n] \to \mathbb{C}[S_n]$.

This map is a projection $\mathbb{C}[S_n] \to V_\lambda$ so its trace is $\dim(V_\lambda) = \mathrm{trace}(\frac{1}{K} L) = \frac{1}{K} \mathrm{trace}(L)$.
As already know that $\mathrm{trace}(L) = n!$ we conclude that $\frac{1}{K} n! = \dim(V_\lambda)$ so $K = \frac{n!}{\dim(V_\lambda)}$. □

Suppose A is an algebra with an idempotent $e = e^2 \in A$.
Let M be a left A-module and for each $m \in eM$ let $\rho_m : Ae \to M$ be the map with $\rho_m(a) = am$.

Lemma. The operations $m \mapsto \rho_m$ and $f \mapsto f(e)$ are inverse bijections $eM \cong \mathrm{Hom}_A(Ae, M)$.

Proof. It is clear that if $m \in eM$ then $\rho_m \in \mathrm{Hom}_A(Ae, M)$.
Also notice that if $f \in \mathrm{Hom}_A(Ae, M)$ then $f(e) = f(e^2) = ef(e) \in eM$.

Moreover, if $a \in Ae$ then $ae = a$ so $\rho_{f(e)}(a) = af(e) = f(ae) = f(a)$ so $\rho_{f(e)} = f$.
Finally, if $m \in eM$ then $em = m$ so $\rho_m(e) = m$. □

Proof of classification theorem for irreducible S_n-modules. Suppose $\lambda, \mu \vdash n$ are such that $\lambda \geq \mu$.

The previous two lemmas with $A = \mathbb{C}[S_n]$ and $e = \frac{\dim V_\lambda}{n!} c_\lambda$ and $M = V_\mu$ imply that

$$\mathrm{Hom}_{\mathbb{C}[S_n]}(V_\lambda, V_\mu) = \mathrm{Hom}_{\mathbb{C}[S_n]}(\mathbb{C}[S_n]c_\lambda, \mathbb{C}[S_n]c_\mu) \cong c_\lambda \mathbb{C}[S_n] c_\mu \cong \begin{cases} \mathbb{C}c_\lambda & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda > \mu. \end{cases}$$

Therefore this Hom-space is 1-dimensional if $\lambda = \mu$ and zero otherwise.

By Schur's lemma this means that V_λ is irreducible and $V_\lambda \not\cong V_\mu$ if $\lambda \neq \mu$.

Finally, observe that the number of partitions of n is the same as the number of conjugacy classes on S_n, which is also the number of distinct isomorphism classes of irreducible S_n-representations.

Thus the V_λ's give all isomorphism classes of irreducible complex S_n-representations □