1   Review: complex representations of symmetric groups

Last time we constructed all complex irreducible representations of the finite symmetric groups.

Here is a quick review.

Let n be a positive integer. Define S_n to be the group of permutations of $[n] = \{1, 2, \cdots, n\}$.

For each partition $\lambda = (\lambda_1, \cdots, \lambda_\ell) \vdash n$, there is an associated *Specht module* constructed as follows:

$$V_\lambda = \mathbb{C}[S_n] a_\lambda b_\lambda$$

where

$$a_\lambda = \sum_{g \in \text{row stabilizer of } T_\lambda} g \quad \text{and} \quad b_\lambda = \sum_{g \in \text{column stabilizer of } T_\lambda} \text{sgn}(g) g.$$

with T_λ the unique standard tableau whose rows are all consecutive, so that

$$T_{(3,3,1,1)} = \begin{array}{|c|c|c|}
\hline
1 & 2 & 3 \\
\hline
4 & 5 & 6 \\
\hline
7 \\
\cline{1-1}
8 \\
\cline{1-1}
\end{array}.$$

The *row stabilizer* contains all $\sigma \in S_n$ with $\sigma(i) = j$ only when i and j are in the same row of T_λ.

The *column stabilizer* contains all $\sigma \in S_n$ with $\sigma(i) = j$ only when i and j are in the same column of T_λ.

Theorem. Each V_λ for $\lambda \vdash n$ is an irreducible complex representation of S_n.

In addition, each irreducible complex representation of S_n is isomorphic to V_λ for a unique $\lambda \vdash n$.

Example. The trivial representation of S_n is V_λ for $\lambda = (n)$. The sign

The sign representation of S_n is V_λ for $\lambda = (1, 1, \ldots, 1)$.

We will see next time that these are the only 1-dimensional representations of S_n up to isomorphism.

Remark. For any finite group, there is a bijection between the set of conjugacy classes and isomorphism classes of irreducible complex representations (because the character table is a square matrix).

For S_n, we can specify this bijection in a concrete way: $\Big\{\text{permutations of cycle type } \lambda\Big\} \leftrightarrow V_\lambda$.

Recall that the *cycle type* of a permutation like $\sigma = (1)(2\ 5\ 7\ 8)(3)(4\ 9\ 6)$ is $\lambda = (4, 3, 1, 1)$.

2   Schur-Weyl duality

Schur-Weyl duality is a fundamental relationship between the irreducible representations of symmetric groups and general linear groups.

For the rest of this lecture, all algebras and representations are defined over an algebraically closed field.

Let E be a finite-dimensional vector space.

Let A and B be subalgebras of the endomorphism algebra $\mathsf{End}(E) = \{$ linear maps $E \to E$ $\}$.

Assume A is semisimple and $B = \{b \in \mathsf{End}(E) : ab = ba \text{ for all } a \in A\}$.

In other words $B = \mathsf{End}_A(E)$ is the space of A-representation morphisms from E to itself.

Theorem (*Double centralizer theorem*). In this setup, the following properties hold:

(1) $A = \{a \in \mathsf{End}(E) : ab = ba \text{ for all } b \in B\} = \mathsf{End}_B(E)$.

   (To indicate this situation, we say that A and B are *commuting algebras* of each other.)

(2) B is also semisimple.

(3) E is a representation of $A \otimes B$ under the linear action
$$a \otimes b : e \mapsto a(b(e)) = b(a(e)) \qquad \text{for } a \in A, \, b \in B \text{ and } e \in E.$$

(4) There are irreducible representations $\{V_i\}_{i \in I}$ of A and $\{W_i\}_{i \in I}$ of B such that
$$E \cong \bigoplus_{i \in I} V_i \otimes W_i$$

   as an $A \otimes B$ representation. These representations have the same index set and $\boxed{W_i \cong \mathrm{Hom}_A(V_i, E)}$.

(5) Each irreducible representation of A is isomorphic to V_i for a **unique** index $i \in I$.

   Each irreducible representation of B is likewise isomorphic to W_i for a **unique** index $i \in I$.

   Thus, the map
$$V_i \leftrightarrow W_i$$

   gives a bijection between the isomorphism classes of irreducible A- and B-representations.

   We refer to this bijection as the *correspondence* defined by E.

Proof. Since A is semisimple, we have
$$A \cong \bigoplus_{i \in I} \mathsf{End}(V_i) \quad \text{and} \quad E \cong \bigoplus_{i \in I} V_i \otimes W_i$$

where $\{V_i\}_{i \in I}$ represent the distinct $\cong$-classes of irreducible A-representations and $W_i \cong \mathrm{Hom}_A(V_i, E)$. Once we make these identifications, Schur's lemma tells us that $B \cong \bigoplus_{i \in I} \mathsf{End}(W_i)$.

This implies that B is semisimple.

Schur's lemma then implies that A is the commuting algebra of B in $\mathsf{End}(E)$.

The remaining assertions follow by writing
$$A = \bigoplus_{i \in I} \mathsf{End}(V_i), \qquad E = \bigoplus_{i \in I} V_i \otimes W_i, \qquad \text{and} \qquad B = \bigoplus_{i \in I} \mathsf{End}(W_i).$$

$\square$

From this point on, for simplicity, we assume the ambient field is $\mathbb{C}$.

Choose a nonzero finite-dimensional $\mathbb{C}$-vector space V and a positive integer n.

Let $\mathrm{GL}(V)$ be the group of invertible linear maps $V \to V$.

We now focus on the vector space $V^{\otimes n}$ which has dimension $(\dim V)^n$.

The vector space $\mathsf{End}(V^{\otimes n})$ of all linear maps $V^{\otimes n} \to V^{\otimes n}$ has dimension $(\dim V)^{2n}$.

Let $\Phi : \mathsf{End}(V)^{\otimes n} \to \mathsf{End}(V^{\otimes n})$ be the linear map that satisfies

$$\Phi(f_1 \otimes f_2 \otimes \cdots \otimes f_n)(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = f_1(v_1) \otimes f_2(v_2) \otimes \cdots \otimes f_n(v_n).$$

for all $f_i \in \mathsf{End}(V)$ and $v_i \in V$.

Lemma. The map Φ is an isomorphism of vector spaces.

Proof. Both $\mathsf{End}(V)^{\otimes n}$ and $\mathsf{End}(V^{\otimes n})$ have the same finite dimension.

So we just need to check that Φ is surjective.

Choose a basis $v_1, v_2, \ldots, v_m$ for V. Then the elements $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n}$ are a basis for $V^{\otimes n}$.

Let $E_{ij} \in \mathsf{End}(V)$ be the linear map that sends $v_j \mapsto v_i$ and all other basis elements of V to zero.

Similarly let $E_{(i_1,\ldots,i_n),(j_1,\ldots,j_n)} \in \mathsf{End}(V^{\otimes n})$ be the linear map that sends

$$v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_n} \mapsto v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n}$$

while sending all other basis elements of $V^{\otimes n}$ to zero.

Then the maps E_{ij} are a basis for $\mathsf{End}(V)$ and the maps $E_{(i_1,\ldots,i_n),(j_1,\ldots,j_n)}$ are a basis for $\mathsf{End}(V^{\otimes n})$.

But we have $E_{(i_1,\ldots,i_n),(j_1,\ldots,j_n)} = \Phi(E_{i_1 j_1} \otimes E_{i_2 j_2} \otimes \cdots E_{i_n j_n})$.

Thus a basis for $\mathsf{End}(V^{\otimes n})$ belongs to the image of $\Phi : \mathsf{End}(V)^{\otimes n} \to \mathsf{End}(V^{\otimes n})$.

Therefore Φ is surjective since it is linear map. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Going forward, we identify $\mathsf{End}(V^{\otimes n})$ with $\mathsf{End}(V)^{\otimes n}$ via the map Φ.

So rather than writing $\Phi(f_1 \otimes f_2 \otimes \cdots \otimes f_n) \in \mathsf{End}(V^{\otimes n})$ when $f_1, f_2, \ldots, f_n \in \mathsf{End}(V)$, we just write

$$f_1 \otimes f_2 \otimes \cdots \otimes f_n \in \mathsf{End}(V^{\otimes n}).$$

There is a unique way to make $V^{\otimes n}$ into an S_n-representation in which $\sigma \in S_n$ acts on pure tensors as

$$\sigma(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.$$

In this representation, any transposition $(i \ j) \in S_n$ acts by swapping the tensor factors v_i and v_j.

Example. To see why we need the inverse in the definition of this representation, suppose $n = 3$.

Let $s_1 = (1\ 2) = 213 \in S_3$ and $s_2 = (2\ 3) = 132 \in S_3$ and choose any $v_1, v_2, v_3 \in V$. Then we have

$$s_2(v_1 \otimes v_2 \otimes v_3) = v_1 \otimes v_3 \otimes v_2 \qquad \text{and} \qquad s_1(v_1 \otimes v_3 \otimes v_2) = v_3 \otimes v_1 \otimes v_2.$$

As it holds that $s_1 s_2 = (1\ 2\ 3) = 231$, we then have $(s_1 s_2)^{-1} = (1\ 3\ 2) = 312$ so

$$(s_1 s_2)(v_1 \otimes v_2 \otimes v_3) = v_3 \otimes v_1 \otimes v_2 = s_1(s_2(v_1 \otimes v_2 \otimes v_3)).$$

Define $\mathsf{Sym}^n(V)$ to be the subspace of vectors in $V^{\otimes n}$ fixed by all $\sigma \in S_n$ under the action just described.

Recall the definitions of the *tensor algebra* $\mathcal{T}V = \bigoplus_{n \geq 0} V^{\otimes n}$ and the *symmetric algebra*

$$\mathcal{S}V = \mathcal{T}V / \langle v \otimes w - w \otimes v : v, w \in V \rangle.$$

Let $\pi : \mathcal{T}V \to \mathcal{S}V$ be the quotient map and define $S^n V = \pi(V^{\otimes n})$.

The vector spaces $V^{\otimes n}$, $S^n V$, and $\mathsf{Sym}^n V$ are all $\mathrm{GL}(V)$-representations under the action

$$g : v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto gv_1 \otimes gv_2 \otimes \cdots \otimes gv_n \qquad \text{for } g \in \mathrm{GL}(V).$$

Lemma. The quotient map π restricts to an isomorphism of $\mathrm{GL}(V)$-representations $\mathsf{Sym}^n(V) \to S^n V$.

The inverse of this isomorphism is the linear map Ψ that acts on the image of a pure tensor in $S^n V$ as

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}.$$

This lemma holds over more general fields than $\mathbb{C}$ as long as $n!$ is an invertible scalar.

Proof. It is clear that Ψ is a well-defined map $S^n V \to \mathsf{Sym}^n(V)$ such that $\pi \circ \Psi = \mathrm{id}$.

If $X \in \mathsf{Sym}^n(V) \subset V^{\otimes n}$ then then $\Psi \circ \pi(X) = \frac{1}{n!} \sum_{\sigma \in S_n} X = X$.

Thus $\pi : \mathsf{Sym}^n(V) \to S^n V$ and $\Psi : S^n V \to \mathsf{Sym}^n(V)$ are inverse isomorphisms of vector spaces.

Both maps commute with the action of $\mathrm{GL}(V)$. $\hfill \square$

Lemma. The vector space $\mathsf{Sym}^n(V)$ is spanned by elements of the form $v \otimes v \otimes \cdots \otimes v$ for $v \in V$.

Proof. A homework exercise showed that $S^n V$ is an irreducible $\mathrm{GL}(V)$-representation.

Thus $\mathsf{Sym}^n(V) \cong S^n V$ is also an irreducible $\mathrm{GL}(V)$-representation by the previous lemma.

The subspace spanned by all $v \otimes \cdots \otimes v$ is a nonzero subrepresentation, so it must be equal to $\mathsf{Sym}^n(V)$. $\hfill \square$

Now let A be the image of $\mathbb{C}[S_n]$ in $\mathsf{End}(V^{\otimes n})$ relative to the S_n-representation defined above.

Next, define $B = \mathsf{End}_A(V^{\otimes n}) = \{b \in \mathsf{End}(V^{\otimes n}) : ab = ba \text{ for all } a \in A\}$.

Proposition. The vector space B is spanned by the elements $f \otimes f \otimes \cdots \otimes f$ for all $f \in \mathsf{End}(V)$.

Proof. Define an S_n-representation on $\mathsf{End}(V)^{\otimes n} = \mathsf{End}(V^{\otimes n})$ as above, just replacing V by $\mathsf{End}(V)$.

This means that $\sigma \in S_n$ acts on a pure tensor by the formula

$$\sigma(f_1 \otimes \cdots \otimes f_n) = f_{\sigma^{-1}(1)} \otimes \cdots \otimes f_{\sigma^{-1}(n)} \qquad \text{for } f_1, \ldots, f_n \in \mathsf{End}(V).$$

Observe that if $f_1, \ldots, f_n \in \mathsf{End}(V)$ and $v_1, \ldots, v_n \in V$ and $\sigma \in S_n$ then

$$\sigma \circ (f_1 \otimes \cdots \otimes f_n) \circ \sigma^{-1}(v_1 \otimes \cdots \otimes v_n) = \sigma \circ (f_1 \otimes \cdots \otimes f_n)(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)})$$

$$= \sigma\Big(f_1(v_{\sigma(1)}) \otimes \cdots \otimes f_n(v_{\sigma(n)})\Big)$$

$$= w_{\sigma^{-1}(1)} \otimes \cdots \otimes w_{\sigma^{-1}(n)} \qquad \text{where } w_i = f_i(v_{\sigma(i)})$$

$$= f_{\sigma^{-1}(1)}(v_1) \otimes \cdots \otimes f_{\sigma^{-1}(n)}(v_n).$$

In other words, we have $\sigma \circ (f_1 \otimes \cdots \otimes f_n) \circ \sigma^{-1} = f_{\sigma^{-1}(1)} \otimes \cdots \otimes f_{\sigma^{-1}(n)} = \sigma(f_1 \otimes \cdots \otimes f_n)$.

But an element $f \in \mathsf{End}(V)^{\otimes n}$ belongs to B if and only if $\sigma \circ f \circ \sigma^{-1} = f$ for all $\sigma \in S_n$.

Thus $B = \{f \in \mathsf{End}(V)^{\otimes n} : \sigma(f) = f \text{ for all } \sigma \in S_n\} = \mathsf{Sym}^n(\mathsf{End}(V))$.

The result is therefore immediate by the previous lemma. $\hfill \square$

Proposition. The image of $\mathbb{C}[\mathrm{GL}(V)]$ in $\mathsf{End}(V^{\otimes n})$ is equal to B.

Proof. Denote the span of the images of $g \otimes g \otimes \cdots \otimes g$ in $\mathsf{End}(V^{\otimes n})$ for $g \in \mathrm{GL}(V)$ by B'.

Let $b \in \mathsf{End}(V)$ be any element. We claim that B' contains $b^{\otimes n} = b \otimes b \otimes \cdots \otimes b$.

This holds since for sufficiently small $\epsilon > 0$, we have $\epsilon I + b \in \mathrm{GL}(V)$ so $(\epsilon I + b)^{\otimes n} \in B$.

This implies that $(\epsilon I + b)^{\otimes n} \in B$ for all ϵ since $(\epsilon I + b)^{\otimes n}$ is a polynomial in ϵ. Now set $\epsilon = 0$. $\square$

The algebra A is semisimple since Maschke's theorem implies that its regular representation is semisimple. As we have just shown that $B = \mathsf{End}_A(V^{\otimes n})$, the double centralizer theorem implies the following result:

Theorem (*Schur-Weyl duality*)**.** As $S_n \times \mathrm{GL}(V)$-representations, we have:

$$V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} V_\lambda \otimes L_\lambda$$

where each

$$L_\lambda = \mathrm{Hom}_{S_n}(V_\lambda, V^{\otimes n})$$

is either zero or an irreducible complex representation of $\mathrm{GL}(V)$ corresponding to the Specht module V_λ. Moreover, we have $L_\lambda \not\cong L_\mu$ for non-zero representations with $\lambda \neq \mu$.

Schur-Weyl duality gives a canonical correspondence between irreducible S_n-representations and $\mathrm{GL}(V)$-representations, providing a consistent way of indexing both by integer partitions.

Next time we will discuss explicit character formulas for S_λ and L_λ.

We will see that $L_\lambda \neq 0$ if and only if λ has at most $\dim(V)$ nonzero parts.