1 Review: Schur–Weyl duality

Schur-Weyl duality is a correspondence between between irreducible representations of symmetric groups and general linear groups that explains why both families are indexed by partitions.

Such a duality exists for any pair of *commuting algebras* $A, B \subset \mathsf{End}(V)$ with $\dim(V) < \infty$ and

$$A = \{a \in \mathsf{End}(V) : ab = ba \text{ for all } b \in B\} \quad \text{and} \quad B = \{b \in \mathsf{End}(V) : ab = ba \text{ for all } a \in A\},$$

as long as one algebra is semisimple (in which case the other will also be semisimple).

The specific instance of this duality that is of interest to us takes the following form.

Fix positive integers n and N. Let $V = \mathbb{C}^N$ and set $\mathrm{GL}_N = \mathrm{GL}(V)$.

Then $V^{\otimes n} = V \otimes \cdots \otimes V$ is an S_n- and GL_N-representation in which $\sigma \in S_n$ and $g \in \mathrm{GL}_N$ act as

$$\sigma(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(n)} \quad \text{and} \quad g(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_n.$$

Let A and B be the images of $\mathbb{C}[S_n]$ and $\mathbb{C}[\mathrm{GL}_N]$ in $\mathsf{End}(V^{\otimes n})$ defined by these representations.

In this setup, *Schur-Weyl duality* refers to the following properties:

(a) A and B are commuting algebras of each other.

(b) A and B are both semisimple, and $V^{\otimes n}$ is a semisimple $\mathbb{C}[S_n] \otimes \mathbb{C}[\mathrm{GL}_N]$-representation.

(c) As a $\mathbb{C}[S_n] \otimes \mathbb{C}[\mathrm{GL}_N]$-representation, $V^{\otimes n}$ decomposes as

$$V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} V_\lambda \otimes L_\lambda,$$

where V_λ is the usual *Specht module* of S_n and

$$L_\lambda = \mathrm{Hom}_A(V_\lambda, V^{\otimes n})$$

is zero or an irreducible GL_N-representation. Also, $L_\lambda \not\cong L_\mu$ if $L_\lambda \neq 0$ and $L_\mu \neq 0$ and $\lambda \neq \mu$.

2 Frobenius character formula

Recall that a *cycle* of $\sigma \in S_n$ is a set of the form $\{\sigma^k(i) : k = 0, 1, 2, \cdots\}$ for some $i \in [n] = \{1, 2, \ldots, n\}$.

The set $[n]$ is a disjoint union of cycles for a fixed permutation $\sigma \in S_n$.

The sizes of these cycles, arranged in descending order, give the *cycle type* of σ, which is a partition of n.

Two permutations in S_n are conjugate if and only if they have the same cycle type.

Let $\mathbf{i} = (i_1, i_2, i_3, \cdots)$ be a sequence of non-negative integers with $n = \sum_{m \geq 1} m \cdot i_m$.

Then choose a permutation $c_{\mathbf{i}} \in S_n$ that has i_m cycles of size m for each $m = 1, 2, 3, \cdots$.

For example, if $i = (2, 1, 1, 0, 0, \cdots)$ then $n = 7$ and $c_{\mathbf{i}}$ could be $(1)(2)(3\ 4)(5\ 6\ 7)$ or $(1\ 4)(2)(3\ 5\ 7)(6)$.

The elements $c_{\mathbf{i}}$ represent the distinct conjugacy classes in S_n.

Choose a partition $\lambda \vdash n$ and let $\ell(\lambda)$ be its number of nonzero parts. Note that $\ell(\lambda) \leq n$.

Theorem (*Frobenius character formula*)**.** Choose any integer $N \geq \ell(\lambda)$. Then the value of character χ_λ of the Specht module V_λ at the permutation $c_\mathbf{i}$ is the coefficient of the monomial:

$$x^{\lambda+(N-1,N-2,\cdots,3,2,1)} = \prod_{j=1}^{N} x_j^{\lambda_j+N-j},$$

in the polynomial $\boxed{\prod_{1\leq j<k\leq N}(x_j - x_k) \cdot \prod_{m\geq 1}(x_1^m + x_2^m + \cdots + x_N^m)^{i_m}}$.

Proof sketch. Call the class function defined by this formula $\theta_\lambda : S_n \to \mathbb{Z}$.

It is possible with some algebraic identities to check that $\theta_\lambda(1) > 0$ and $(\theta_\lambda, \theta_\lambda) = 1$.

This implies that θ_λ is an irreducible character of S_n and hence equals to some χ_μ.

To show that $\theta_\lambda = \chi_\lambda$, one argues that θ_λ has a triangular expansion

$$\theta_\lambda = \chi_\lambda + (\text{ terms } \chi_\mu \text{ with } \mu < \lambda \text{ in lexicographic order}),$$

where the extra terms turn out to be zero, by expressing θ_λ as a $\mathbb{Z}$-linear combination of certain induced characters $\mathrm{Ind}_{P_\lambda}^{S_n}(\mathbf{1})$ whose irreducible decompositions are easy to understand. □

3 Hook length formula

The Frobenius character formula is somewhat unwieldy to calculate and difficult to remember.

However, specialized to $\mathbf{i} = (n, 0, 0, 0, \dots)$, it reduces to a simple and explicit formula for $\chi_\lambda(1) = \dim(V_\lambda)$.

When $\mathbf{i} = (n, 0, 0, 0, \dots)$ the Frobenius character formula tells that $\chi_\lambda(1)$ is equal to the coefficient of

$$x_1^{\lambda_1+N-1} x_2^{\lambda_2+N-2} \cdots x_N^{\lambda_N}$$

in the product

$$(x_1 + x_2 + \cdots + x_N)^n \prod_{1\leq j<k\leq N}(x_j - x_k).$$

Recall the Vandemonde determinant formula

$$\prod_{1\leq j<k\leq N}(x_j - x_k) = \det \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{N-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{N-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_N & x_N^2 & \dots & x_N^{N-1} \end{bmatrix}.$$

Thus, writing $\ell_j = \lambda_j + N - j$ we get

$$\chi_\lambda(1) = \sum_{\substack{\sigma\in S_N \\ \ell_j \geq N-\sigma(j)\ \forall j}} \mathrm{sgn}(\sigma) \cdot \frac{n!}{(\ell_1 - N + \sigma(1))! \cdot (\ell_2 - N + \sigma(2))! \cdot (\ell_3 - N + \sigma(3))! \cdots}.$$

Then one argues that this expression can be rewritten as

$$\frac{n!}{\prod_j \ell_j!} \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) \prod_j \ell_j(\ell_j - 1)\cdots(\ell_j - N + \sigma(j) + 1) = \frac{n!}{\prod_j \ell_j!} \det\left[\ell_j(\ell_j - 1)\cdots(\ell_j - N + i + 1)\right]_{1\leq i,j\leq N}.$$

Finally, using column reduction, we can rewrite the last expression as

$$\chi_\lambda(1) = \frac{n!}{\prod_j \ell_j!} \det[\ell_j^{N-i}]_{1\le i,j\le N} = \frac{n!}{\prod_j \ell_j!} \prod_{1\le j<k\le N} (\ell_j - \ell_k).$$

Let us introduce some notation to summarize this computation.

For each (i,j) in the diagram D_λ, define $h_\lambda(i,j)$ to be the number of positions $(x,y) \in \mathsf{D}_\lambda$ such that

$$x = i \text{ and } j \le y \qquad \text{or} \qquad x \ge i \text{ and } j = y.$$

For example, if $\lambda = (5,4,4,1)$ then $h_\lambda(2,2) = 4$ counts the positions marked in

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Theorem (*Hook length formula*)**.** If $\lambda \vdash n$ then $\boxed{\dim(V_\lambda) = \chi_\lambda(1) = \frac{n!}{\prod_{(i,j)\in D_\lambda} h_\lambda(i,j)}}$.

Proof sketch. Continuing from the formula

$$\chi_\lambda(1) = \frac{n!}{\prod_j \ell_j!} \prod_{1\le j<k\le N} (\ell_j - \ell_k),$$

one just needs to check for each $x = 1,2,3,\cdots$ that

$$\frac{\ell_x!}{\prod_{x<j\le N}(\ell_x - \ell_j)} = \prod_{k=1}^{\lambda_x} h_\lambda(x,k).$$

□

Example. If $\lambda = (5,4,4,1)$ then we can exhibit all values of $h_\lambda(i,j)$ as

8	6	5	4	1
6	4	3	2	
5	3	2	1	
1				

and so $\dim(V_\lambda) = \dfrac{14!}{8\cdot 6\cdot 6\cdot 5\cdot 5\cdot 4\cdot 4\cdot 3\cdot 3\cdot 2\cdot 2} = 7\cdot 7\cdot 13\cdot 11\cdot 3 = 21021.$

4 Schur polynomials

We now present some formulas for the character of the representations L_λ of $\mathrm{GL}_N = \mathrm{GL}_N(\mathbb{C})$.

Let λ be a partition with $\ell(\lambda) \le N$. Define

$$\boxed{\Delta_\lambda(x_1, x_2, \dots, x_N) = \det\left[x_i^{N-j+\lambda_j}\right]_{1\le i,j\le N}}$$

so that $\Delta_\emptyset(x_1,x_2,\dots,x_N) = \det\left[x_i^{N-j}\right]_{1\le i,j\le N} = \prod_{1\le i<j\le N}(x_i - x_j)$ when $\emptyset$ is the empty partition.

Definition. The *Schur polynomial* of λ is the rational function

$$s_\lambda(x_1, x_2, \dots, x_N) = \frac{\Delta_\lambda(x_1, x_2, \dots, x_N)}{\Delta_\emptyset(x_1, x_2, \dots, x_N)} \in \mathbb{Q}(x_1, x_2, \dots, x_N).$$

Proposition. Each $s_\lambda(x_1, x_2, \dots, x_N) \in \mathbb{Z}[x_1, x_2, \dots, x_N]$ is actually a polynomial that is *symmetric* in the sense that it is invariant under all reorderings of the x-variables.

Proof. We claim that the polynomial $\Delta_\emptyset(x_1, x_2, \dots, x_N)$ divides $\Delta_\lambda(x_1, x_2, \dots, x_N)$.

This holds since $\Delta_\lambda(x_1, x_2, \dots, x_N)$ is divisible by each factor $x_i - x_j$ with $1 \le i < j \le N$.

To see this, note that if $x_i = x_j$ then $\Delta_\lambda = 0$ is a determinant of a matrix with two equal rows.

The symmetry of s_λ follows by noting that reordering the variables $x_1, x_2, \dots, x_N$ multiplies the value

$$\Delta_\emptyset(x_1, x_2, \dots, x_N) \qquad \text{and} \qquad \Delta_\lambda(x_1, x_2, \dots, x_N)$$

by the same ± 1 factor, corresponding to the sign of the permutation defined by the reordering.

These factors cancel in the fraction defining s_λ. □

Example. Suppose $\lambda = (3)$ so that $\ell(\lambda) = 1$. Take $N = 2$. Then

$$s_{(3)}(x_1, x_2) = \frac{\det\left[x_i^{N-j+\lambda_j}\right]}{\det\left[x_j^{N-j}\right]} = \frac{\det\begin{bmatrix} x_1^4 & 1 \\ x_2^4 & 1 \end{bmatrix}}{\det\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix}} = \frac{x_1^4 - x_2^4}{x_1 - x_2} = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3$$

One can check that $\boxed{s_{(n)}(x_1, x_2, \dots x_N) = \sum_{1 \le i_1 \le i_2 \le \dots \le i_n \le N} \ x_{i_1} x_{i_2} \dots x_{i_N}}$.

Example. Suppose $\lambda = (1, 1, 1)$ so that $\ell(\lambda) = 3$. Take $N = 3$. Then

$$s_{(1,1,1)}(x_1, x_2, x_3) = \frac{\det\begin{bmatrix} x_1^3 & x_1^2 & x_1 \\ x_2^3 & x_2^2 & x_2 \\ x_3^3 & x_3^2 & x_3 \end{bmatrix}}{\det\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{bmatrix}} = x_1 x_2 x_3$$

One can check that $\boxed{s_{(1,1,\dots,1)}(x_1, x_2, \dots x_N) = \sum_{1 \le i_1 < i_2 < i_3 < \dots < i_n \le N} x_{i_1} x_{i_2} \dots x_{i_n}}$.

Remark. There is a general monomial-positive formula (which we won't prove here)

$$s_\lambda(x_1, x_2, \dots, x_N) = \sum_T x^T \in \mathbb{N}[x_1, x_2, \dots, x_N]$$

where T varies over all *semistandard tableaux* of shape λ with entries contained in $\{1, 2, 3, \dots, N\}$, and

$$x^T = \prod_{i=1}^N x_i^{\#(\text{entries of } T \text{ equal to i})}.$$

A tableau is *semistandard* if its rows are weakly increasing and its columns are strictly increasing.

5 Weyl character formula

Again let $\mathbf{i} = (i_1, i_2, i_3, \cdots)$ be a sequence of non-negative integers with $n = \sum_{m\geq 1} m \cdot i_m$.

Then choose a permutation $c_\mathbf{i} \in S_n$ that has i_m cycles of size m for each $m = 1, 2, 3, \cdots$.

The following is a consequence of the Frobenius character formula. For a detailed proof, see the textbook.

Proposition. It holds that

$$\prod_{m\geq 1} (x_1^m + x_2^m + \cdots + x_N^m)^{i_m} = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)\leq N}} \chi_\lambda(c_\mathbf{i}) s_\lambda(x_1, x_2, \ldots, x_N)$$

where χ_λ is the character of the Specht module V_λ.

Recall from Schur-Weyl duality that we have representations L_λ indexed by partitions λ for the group

$$\mathrm{GL}_N = \mathrm{GL}_N(\mathbb{C}) = \mathrm{GL}(\mathbb{C}^N)$$

Each L_λ is either zero or irreducible.

Let $g \in GL_N$ and suppose the eigenvalues of g are $x_1, x_2, \ldots, x_N$.

Theorem (*Weyl character formula*)**.** The GL_N-representation L_λ is nonzero if and only if $\ell(\lambda) \leq N$.

Assume $\ell(\lambda) \leq N$. Then the value of the character of L_λ at g is $\boxed{s_\lambda(x_1, x_2, \ldots x_N) \in \mathbb{C}}$.

In particular, $\boxed{\dim(L_\lambda) = s_\lambda(1, 1, \ldots, 1) = \prod_{1\leq i<j\leq N} \frac{\lambda_i - \lambda_j + j - i}{j - i}} \in \mathbb{N}$.

Proof sketch. See the textbook for a detailed argument.

This idea is to directly compute the trace of $g^{\otimes n} c_\mathbf{i}$ acting in $(\mathbb{C}^N)^{\otimes n}$, and note that this is equal to

$$\textstyle\sum_\lambda \chi_\lambda(c_\mathbf{i}) \mathrm{trace}_{L_\lambda}(g)$$

by Schur–Weyl duality. The dimension formulas is a special case of a more general algebraic identity. □

Remark. Recall that the Specht modules V_λ give all irreducible complex representations of S_n.

It is shown in the textbook that the representations L_λ for partitions λ with $\ell(\lambda) \leq N$ give the slightly smaller family of all irreducible *polynomial representations* of GL_N.

A *polynomial representation* of GL_N is a finite-dimensional complex representation (ρ, V) where is some basis of V, all entries in the matrix of $\rho(g)$ are polynomial functions in the entries of g and $1/\det(g)$.

Each L_λ is a polynomial representation, as it is subrepresentation of $(\mathbb{C}^N)^{\otimes n}$.

Showing that any irreducible polynomial representation is isomorphic to some L_λ takes more work.