Instructions: Complete the following exercises.

Solutions must be hand-written and submitted in-person.

You will be graded on clarity and simplicity as well as correctness.

You may use any resources and work with other students, but you must write up your own solutions.

Due on **Tuesday, February 24**.

1. Let A be an $\mathbb{F}$-algebra, not necessarily associative. Show that the vector space of derivations $\mathsf{Der}(A)$ is a Lie subalgebra of $\mathfrak{gl}(A)$. (See §III.1.3 in the textbook.)
2. Show that there exists a unique 2-dimensional Lie algebra $L = \mathbb{F}\text{-span}\{X, Y\}$ with $[X, Y] = X$. Find a subalgebra of $\mathfrak{gl}_n(\mathbb{F})$ for some n that is isomorphic to L. Finally, show that L is solvable but not nilpotent.
3. Suppose $X \in \mathfrak{gl}_n(\mathbb{F})$ has n distinct eigenvalues $a_1, a_2, \ldots, a_n \in \mathbb{F}$. Prove that the eigenvalues of ad_X are the n^2 scalars $a_i - a_j$ for $1 \leq i, j \leq n$ (which are not necessarily distinct).
4. Let L be a Lie algebra over an algebraically closed field and let $X \in L$. Prove that the subspace of L spanned by the eigenvectors of ad_X is a Lie subalgebra.
5. Show that $[\mathfrak{sl}_n(\mathbb{F}), \mathfrak{sl}_n(\mathbb{F})] = \mathfrak{sl}_n(\mathbb{F})$ if $\mathbb{F}$ has characteristic zero. Check that $\mathfrak{sl}_2(\mathbb{F})$ is nilpotent if $\mathbb{F}$ has characteristic 2.
6. Prove that a Lie algebra L is solvable if and only if there exists a chain of Lie subalgebras $L = L_0 \supset L_1 \supset L_2 \supset \cdots \supset L_k = 0$ such that each L_{i+1} is an ideal of L_i with L_i/L_{i+1} abelian.
7. Let L be a nilpotent Lie algebra. Prove that L has an ideal of codimension 1.
8. Prove that there exists a non-degenerate alternating bilinear form B on a finite-dimensional $\mathbb{F}$-vector space V if and only if $\dim(V)$ is even. When B is such a form and $n = \dim(V)$, show that the Lie algebra $L_B \subseteq \mathfrak{gl}(V)$ from Lecture 1 is isomorphic to $\mathfrak{sp}_n(\mathbb{F})$.

   Alternating means that $B(v, v) = 0$ for all $v \in V$.

   Non-degenerate means that if $B(v, w) = 0$ for all $w \in V$ then $v = 0$.

   Be sure that your argment works even when $\mathrm{char}(\mathbb{F}) = 2$.
9. Assume $\mathrm{char}(\mathbb{F}) \neq 2$ and $\mathbb{F}$ is algebraically closed. Suppose B is a non-degenerate symmetric bilinear form on a finite-dimensional $\mathbb{F}$-vector space V with $n = \dim(V)$. Prove that $L_B \cong \mathfrak{o}_n(\mathbb{F})$.
10. Assume $\mathrm{char}(\mathbb{F}) = 2$ and $\mathbb{F}$ is algebraically closed (so that when n is even we have $\mathfrak{o}_n(\mathbb{F}) = \mathfrak{sp}_n(\mathbb{F})$). Suppose B is a non-degenerate bilinear form on a finite-dimensional $\mathbb{F}$-vector space V. Show that if B is alternating then B is symmetric. When B is symmetric and $n = \dim(V)$, does it always hold that $L_B \cong \mathfrak{o}_n(\mathbb{F})$? Justify your answer by giving a proof or a counterexample.