Instructions: Complete the following exercises.

Solutions must be hand-written and submitted in-person.

You will be graded on clarity and simplicity as well as correctness.

You may use any resources and work with other students, but you must write up your own solutions.

Due on **Tuesday, March 3**.

Suppose V is a finite-dimensional vector space over any algebraically closed field $\mathbb{F}$.

Let $X : V \to V$ be a linear map. The characteristic polynomial *of X is $\det(tI - X) \in \mathbb{F}[t]$. Suppose this polynomial factors as $\det(tI - X) = \prod_{i=1}^{k}(t - a_i)^{m_i}$ where $a_i \neq a_j \in \mathbb{F}$ for $i \neq j$ and $m_i > 0$.*

The following three exercises prove the existence of the generalized eigenspace decomposition corresponding to X and then the Jordan decomposition of X. Your solutions to these problems therefore should only use elementary linear algebra and should not involve the Jordan canonical form, the Cayley–Hamilton theorem, or similar results.

1. Define $V_i = \{v \in V : (X - a_i I)^n v = 0 \text{ for some } n > 0\}$. Show that $V = \bigoplus_{i=1}^{k} V_i$.

   In other words, check that $V_i \cap V_j = 0$ if $i \neq j$ and $V = V_1 + V_2 + \cdots + V_k$.

2. By considering the matrix of X in an appropriate basis, show that the characteristic polynomial of X is the product of the characteristic polynomials of the restricted maps $X|_{V_i} : V_i \to V_i$ for $i = 1, 2, \ldots, k$. Use this to deduce that $\dim(V_i) = m_i$ and that $V_i = \ker(X - a_i I)^{m_i}$.

3. Look up the statement of the Chinese Remainder Theorem for the polynomial ring $\mathbb{F}[t]$ and explain why this implies that there is a polynomial $p(t) \in \mathbb{F}[t]$ with

   $$p(t) \equiv a_i \pmod{(t - a_i)^{m_i}} \text{ for } i = 1, 2, \ldots, k \quad \text{and} \quad p(t) \equiv 0 \pmod{T}.$$

   Show that if $X_{\mathrm{ss}} = p(X)$ and $X_{\mathrm{nil}} = X - X_{\mathrm{ss}}$ then X_{ss} is diagonalizable and X_{nil} is nilpotent.

4. Assume $\mathbb{F}$ has characteristic zero and let $L = \mathfrak{sl}(V)$.

   Use Lie's Theorem to prove that $\mathsf{Rad}(L) = Z(L)$. Then show that $Z(L) = 0$ so L is semsimple.

5. Suppose $X, Y \in \mathfrak{gl}(V)$ commute. Show that $(X + Y)_{\mathrm{ss}} = X_{\mathrm{ss}} + Y_{\mathrm{ss}}$ and $(X + Y)_{\mathrm{nil}} = X_{\mathrm{nil}} + Y_{\mathrm{nil}}$.

   Show by example that this can fail if $XY \neq YX$.

6. Show that if L is a nilpotent Lie algebra then the Killing form is identically zero.

7. Show that a Lie algebra L is solvable if and only if $[L, L]$ is inside the radical of the Killing form.

8. Compute the basis of $\mathfrak{sl}_2(\mathbb{F})$ dual to the standard basis $\{E, F, H\}$, relative to the killing form.

9. Assume L is a finite-dimensional, semisimple Lie algebra defined over an algebraically closed field $\mathbb{F}$ of characteristic zero. Let $L = L_1 \oplus L_2 \oplus \cdots \oplus L_m$ be the decomposition of L into its simple ideals.

   Fix $X \in L$ and let $X_i \in L_i$ be the unique elements such that $X = X_1 + X_2 + \cdots + X_m$.

   Show that the semisimple and nilpotent parts of X are the sums of the semisimple and nilpotent parts of the various components X_i. That is, show that

   $$X_{\mathrm{ss}} = \sum_i (X_i)_{\mathrm{ss}} \quad \text{and} \quad X_{\mathrm{nil}} = \sum_i (X_i)_{\mathrm{nil}}.$$

10. Prove that if L is a solvable Lie algebra defined over an algebraically closed field $\mathbb{F}$ of characteristic zero then every irreducible representation of L is one-dimensional.