Instructions: Complete the following exercises.

Solutions must be hand-written and submitted in-person.

You will be graded on clarity and simplicity as well as correctness.

You may use any resources and work with other students, but you must write up your own solutions.

Due on **Tuesday, March 10**.

Unless mentioned otherwise, any Lie algebras below are finite-dimensional and defined over an algebraically closed field $\mathbb{F}$ of characteristic zero.

1. Using the standard basis for $L = \mathfrak{sl}_2(\mathbb{F})$, write down the Casimir element of the adjoint representation of L.

2. Use Weyl's theorem to prove that $\mathsf{ad}(L) = \mathsf{Der}(L)$ when L is semisimple.

   Recall that $\mathsf{Der}(L)$ is defined to be the set of linear maps $\delta : L \to L$ satisfying

   $$\delta([X, Y]) = [X, \delta(Y)] + [\delta(X), Y].$$

3. A Lie algebra L is *reductive* if $\mathsf{Rad}(L) = Z(L)$. Suppose L is reductive.

   Show that L is a completely reducible $\mathsf{ad}(L)$-module, that $L = Z(L) \oplus [L, L]$ (as Lie algebras), and that $[L, L]$ is semisimple.

4. Let $L = \mathfrak{sl}(V)$ for a finite-dimensional vector space V.

   Use Lie's Theorem to prove that $\mathsf{Rad}(L) = Z(L)$.

   Then show that $Z(L) = 0$ so L is semsimple.

5. Prove Schur's lemma as stated in Lecture 6.

   Then let L be a simple Lie algebra and suppose β and γ are two bilinear forms $L \times L \to \mathbb{F}$ that are nondegenerate, symmetric, and associative.

   Use Schur's Lemma to prove that β is a nonzero scalar multiple of γ.

6. Let m be a nonnegative integer and let $V(m)$ be a vector space with basis $v_0, v_1, v_2, \ldots, v_m$.

   Define $Hv_i = (m - 2i)v_i$ and $Yv_i = (i + 1)v_{i+1}$ and $Xv_i = (m - i + 1)v_{i-1}$ where $v_{-1} = v_{m+1} = 0$.

   Show that these formulas extend to a module structure for the Lie algebra $\mathfrak{sl}_2(\mathbb{F})$ where

   $$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

   To check this, verify that the matrices describing the action of H, F, and E on $V(m)$ satisfy the same Lie bracket equations as H, F, and E do.

7. The Lie algebra $M = \mathfrak{sl}_3(\mathbb{F})$ contains a copy of $\mathfrak{sl}_2(\mathbb{F})$ in its upper left 2×2 position.

   We can view M as an $\mathfrak{sl}_2(\mathbb{F})$-module via the adjoint representation.

   Find nonnegative integers $n_1, n_2, \ldots, n_k$ such that $M \cong V(m_1) \oplus V(m_2) \oplus \cdots \oplus V(m_k)$.

   Prove your answer.

8. Suppose (just for this exercise) that $\mathbb{F}$ has characteristic $p > 0$. What numbers can occur as p? Show that the $\mathfrak{sl}_2(\mathbb{F})$-module $V(m)$ irreducible if $m < p$, but reducible when $m = p$.

9. Let $\lambda \in \mathbb{F}$ be an arbitrary scalar (now with $\mathbb{F}$ having characteristic zero).

   Let $M(\lambda)$ be a vector space with a countably infinite basis $v_0, v_1, v_2, \ldots$.

   Define $Hv_i = (\lambda - 2i)v_i$ and $Fv_i = (i+1)v_{i+1}$ and $Ev_i = (\lambda - i + 1)v_{i-1}$ where $v_{-1} = 0$.

   These formulas make $M(\lambda)$ into an $\mathfrak{sl}_2(\mathbb{F})$-module. (You can assume this without giving a proof.)

   For which values of λ is $M(\lambda)$ irreducible? Prove your answer.