Instructions: Complete the following exercises.

Solutions must be hand-written and submitted in-person.

You will be graded on clarity and simplicity as well as correctness.

You may use any resources and work with other students, but you must write up your own solutions.

Due on **Tuesday, March 31**.

Except when mentioned otherwise, all Lie algebras and vector spaces below are finite-dimensional and defined over an algebraically closed field $\mathbb{F}$ of characteristic zero.

1. Let L be a semisimple Lie algebra with a maximal toral subalgebra H.

   Prove that if $h \in H$ then the centralizer $C_L(h) := \{X \in L : [X, h] = 0\}$ is a reductive Lie algebra.

Recall that if $\Phi \subset E$ and $\Phi' \subset E'$ are root systems, then an isomorphism $\Phi \to \Phi'$ is a linear bijection $f : E \to E'$ with $f(\Phi) = \Phi'$ and $\langle f(\beta), f(\alpha)\rangle = \langle \beta, \alpha\rangle$ for all $\alpha, \beta \in \Phi$, where $\langle \beta, \alpha\rangle = 2(\beta, \alpha)/(\alpha, \alpha)$. The map f does not need to be an isometry with respect to the forms $(\cdot, \cdot)$ on E and E'.

2. Prove that every three dimensional semisimple Lie algebra has the same root system as $\mathfrak{sl}_2(\mathbb{F})$ up to isomorphism, and so is isomorphic to $\mathfrak{sl}_2(\mathbb{F})$.

3. Prove that no four, five, or seven dimensional semisimple Lie algebras exist.

Below, let Φ be a root system in a vector space E with positive definite form $(\cdot, \cdot)$ and Weyl group W.

4. Define $\Phi^\vee = \{\alpha^\vee : \alpha \in \Phi\}$ where $\alpha^\vee = 2\alpha/(\alpha, \alpha)$.

   Check that $\langle \alpha^\vee, \beta^\vee\rangle = \langle \beta, \alpha\rangle$ for $\alpha, \beta \in \Phi$.

   Prove that $\Phi^\vee$ is a root system (in the same vector space) whose Weyl group is isomorphic to W.

   Draw a picture of $\Phi^\vee$ versus Φ when Φ has type A_1, A_2, B_2, and G_2.

5. Let Φ' be a nonempty subset of Φ with $-\Phi' = \Phi'$, such that if $\alpha, \beta \in \Phi'$ and $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Phi'$. Prove that Φ' is a root system in the subspace of E that it spans.

6. Fix a simple system Δ for Φ and define $\Phi^\pm \subset \Phi$ accordingly.

   Prove that there is a unique element $w_0 \in W$ with $w_0(\Phi^+) = \Phi^-$.

   Prove that any reduced expression for w_0 must involve every simple reflection r_α for $\alpha \in \Delta$.

7. Prove that if Φ is irreducible then so is $\Phi^\vee$.

8. Prove that if $0 \neq \alpha \in E$ and the reflection r_α belongs to W, then $r_\alpha = r_\beta$ for some $\beta \in \Phi$.