**Instructions:** Complete the following exercises.

Solutions must be hand-written and submitted in-person.

You will be graded on clarity and simplicity as well as correctness.

You may use any resources and work with other students, but you must write up your own solutions.

Due on **Thursday, April 16**.

*Let $\Lambda$ and $\Lambda^+$ be the sets of* weights *and* dominant weights *for a root system $\Phi$ with Weyl group $W$ as defined in §13 of the textbook. Consult that section of the textbook to do the following exercises.*

1. Suppose $\lambda \in \Lambda^+$ and $\sigma \in W$. Let $\delta = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$ be half the sum of the positive roots in $\Phi$ relative to an arbitrary simple system. Prove that $\sigma(\lambda + \delta) - \delta \in \Lambda^+$ if and only if $\sigma = 1$.
2. Prove that each subset of $\Lambda$ is contained in a unique smallest saturated set (in the sense defined in §13.4), which is finite if the subset in question is finite.

*Below, let $L$ be a Lie algebra and write $\mathfrak{U}(L)$ for its universal enveloping algebra.*

3. Prove that if $\dim L < \infty$ then $\mathfrak{U}(L)$ has no zero divisors.
4. If $X \in L$ then extend $\mathsf{ad}X$ to an endomorphism of $\mathfrak{U}(L)$ by setting $\mathsf{ad}X(Y) = XY - YX$ for $Y \in \mathfrak{U}(L)$. Prove that if $\dim L < \infty$ then each element of $\mathfrak{U}(L)$ belongs to a finite-dimensional $L$-submodule with respect to this adjoint action.
5. If $L$ is a free Lie algebra on a set $X$, prove that $\mathfrak{U}(L)$ is isomorphic to the tensor algebra on a vector space having $X$ as a basis.
6. Describe the free Lie algebra on a set $X = \{x\}$ of size one.