Instructions: Complete the following exercises.

Solutions must be hand-written and submitted in-person.

You will be graded on clarity and simplicity as well as correctness.

You may use any resources and work with other students, but you must write up your own solutions.

Due on **Thursday, May 7**.

Let L denote a semisimple Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic zero, with a chosen Cartan subalgebra H. Write $\Phi \subset H^*$ be the corresponding root system and let $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a simple system for Φ. Let W denote the Weyl group of Φ and let Λ^+ denote the associated set of dominant integral weights in H^*.

1. (a) Show that if V is an arbitrary L-module then the sum of its weight spaces is direct.

   In other words, show that the intersection of any two weight spaces is trivial.

   (b) Next, show that if V is an irreducible L-module with at least one nonzero weight space, then V is equal to the sum of its weight spaces.

2. For each positive integer d, prove that the number of distinct irreducible L-modules $V(\lambda)$ of dimension at most d is finite. Here $V(\lambda)$ denotes the irreducible standard cyclic module of weight $\lambda \in H^*$ described in §20.3 of the textbook.

3. Show that if $\lambda \in \Lambda^+$ then

   (a) $V(\lambda)^* \cong V(-w_0\lambda)$ where $w_0 \in W$ is the element sending all positive roots to negative roots.

   (b) 0 occurs as a weight of $V(\lambda)$ if and only if λ is a sum of positive roots.

4. Use character theory (namely, the results in §22.5 of the textbook) to show that if $L = \mathfrak{sl}_2(\mathbb{F})$ then

$$V(m) \otimes V(n) \cong V(m+n) \oplus V(m+n-2) \oplus \cdots \oplus V(m-n)$$

   for any nonnegative integers $n \le m$. Here $V(m)$ is the irreducible $\mathfrak{sl}_2(\mathbb{F})$-module of dimension $m+1$.

5. Give a direct proof of Weyl's character formula when $L = \mathfrak{sl}_2(\mathbb{F})$.

6. Suppose L is of type A_{n-1} and $\mathbb{K}$ is an extension of the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ (for $p > 0$ prime). Prove that the corresponding Chevalley group $G(\mathbb{K})$ of adjoint type is isomorphic to $\mathsf{PSL}_n(\mathbb{K})$. This group is defined to be the quotient of $\mathsf{SL}_n(\mathbb{K})$ by its center, which consists of the diagonal matrices with all diagonal entries equal to some nth root of unity in $\mathbb{K}$.