1 Course details

We will cover most of Humphrey's textbook *Introduction to Lie algebras and representation theory.*

Some other relevant textbooks are listed on the public course website:

`https://www.math.hkust.edu.hk/~emarberg/teaching/2026/Math5143/`

Grades will be based on (approximately) weekly homework assignments.

All homework assignments must be submitted **in-person, in hand-written form.**

But there will be no exams.

All lectures will be posted on the public course webpage in pdf format (without annotations).

The annotated slides presented in class, which contain the same content, will not be posted.

2 Motivation

This course is about *Lie algebras* and their representations.

Today will cover some basic definitions and important examples.

Very brief motivation: the most interesting groups in physics, geometry, etc., are *Lie groups*, which are groups that are also manifolds in a way that is compatible with the group structure.

Let $\mathbb{F}$ be a field like $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), or $\mathbb{Q}_p$ (p-adic numbers).

Fix a positive integer n.

Common examples of Lie groups:

- *General linear groups:* $\mathrm{GL}_n(\mathbb{F})$ = invertible $n \times n$ matrices over $\mathbb{F}$,
- *Special linear groups:* $\mathrm{SL}_n(\mathbb{F})$ = invertible $n \times n$ matrices M over $\mathbb{F}$ with $\det(M) = 1$.
- *Orthogonal groups:* $\mathrm{O}_n(\mathbb{F})$ = invertible $n \times n$ matrices M over $\mathbb{F}$ with

  $$\langle Mu, Mv\rangle = \langle u, v\rangle \quad \text{for all } u, v \in \mathbb{F}^n,$$

  where $\langle\cdot,\cdot\rangle$ is a symmetric non-degenerate bilinear form.

  A typical choice of $\langle\cdot,\cdot\rangle$ is the *standard form* with

  $$\langle u, v\rangle = u^\top v$$

  and in this case the form is preserved if and only if $M^{-1} = M^\top$.
- *Symplectic groups:* $\mathrm{Sp}_n(\mathbb{F})$ = invertible $n \times n$ matrices M over $\mathbb{F}$ with

  $$\langle Mu, Mv\rangle = \langle u, v\rangle \quad \text{for all } u, v \in \mathbb{F}^n,$$

  where $\langle\cdot,\cdot\rangle$ is a skew-symmetric non-degenerate bilinear form. Such forms exist only if n is even.

The most important features in the geometric and representation theory of Lie groups are controlled by the tangent space at the identity element. This tangent space has more structure than just being a vector space — namely, it is what we will call a *Lie algebra*.

3 Constructive definition of a Lie algebra

What is a Lie algebra?

Let $\mathbb{F}$ be any field. Setting $\mathbb{F} = \mathbb{C}$ is a convenient choice.

Let V be any vector space defined over $\mathbb{F}$.

Write $\mathfrak{gl}(V)$ for the $\mathbb{F}$-vector space of all linear maps $V \to V$.

Remark. Suppose $\dim V = n < \infty$. Also define

$$\mathfrak{gl}_n(\mathbb{F}) = \Big\{ n \times n \text{ matrices over } \mathbb{F} \Big\}.$$

Then $\mathfrak{gl}(V)$ and $\mathfrak{gl}_n(\mathbb{F})$ are isomorphic as vector spaces, but not canonically.

We get an isomorphism for each choice of basis for V.

The isomorphism $\mathfrak{gl}(V) \xrightarrow{\sim} \mathfrak{gl}_n(\mathbb{F})$ corresponding to a given basis $v_1, v_2, \dots, v_n$ is the map assigning

$$\text{each linear operator } X \mapsto \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \text{ (a square matrix)}$$

where $a_{ij} \in \mathbb{F}$ are the coefficients with $X(v_j) = \sum_{i=1}^n a_{ij} v_i$.

When $V = \mathbb{F}^n$ is the space of n-row columns vectors, and $v_1, v_2, \dots, v_n$ is the standard basis, then the isomorphism $\mathfrak{gl}(\mathbb{F}^n) \xrightarrow{\sim} \mathfrak{gl}_n(\mathbb{F})$ is the usual one that assigns to a linear map its *standard matrix*.

We consider $\mathfrak{gl}(\mathbb{F}^n) = \mathfrak{gl}_n(\mathbb{F})$ to be the same object, identified via this isomorphism.

Now given elements $X, Y \in \mathfrak{gl}(V)$ let $[X, Y] = XY - YX$.

Definition. A *Lie algebra* is subspace $L \subseteq \mathfrak{gl}(V)$ such that $[X, Y] \in L$ for all $X, Y \in L$.

The following are Lie algebras according to this definition in case when $V = \mathbb{F}^n$.

- The *general linear Lie algebra* $\mathfrak{gl}_n(\mathbb{F})$.
- $\mathfrak{d}_n(\mathbb{F}) \overset{\text{def}}{=} \{\text{diagonal matrices in } \mathfrak{gl}_n(\mathbb{F})\}$.
- $\mathfrak{t}_n(\mathbb{F}) \overset{\text{def}}{=} \{\text{upper-triangular matrices in } \mathfrak{gl}_n(\mathbb{F})\}$
- $\mathfrak{n}_n(\mathbb{F}) \overset{\text{def}}{=} \{\text{strictly upper-triangular matrices in } \mathfrak{gl}_n(\mathbb{F})\}$.

We can multiply two elements of $\mathfrak{gl}(V)$ by composition.

Any subspace $L \subseteq \mathfrak{gl}(V)$ that is an *algebra* is also a Lie algebra.

An *algebra* means a subspace that contains the product of any two of its elements.

The examples above are all algebras, but not all Lie algebras have this property.

4 Classical examples

Here some examples of Lie algebras $L \subseteq \mathfrak{gl}_n(\mathbb{F}) = \mathfrak{gl}(\mathbb{F}^n)$ that are not algebras:

(A) $\mathfrak{sl}_n(\mathbb{F}) \overset{\text{def}}{=} \{: X \in \mathfrak{gl}_n(\mathbb{F}) : \mathsf{trace}(X) = 0\}$.

Recall that $\mathsf{trace}(X) = X_{11} + X_{22} + \cdots + X_{nn}$ and that $\mathsf{trace}(XY) = \mathsf{trace}(YX)$. Hence

$$\mathsf{trace}([X,Y]) = \mathsf{trace}(XY) - \mathsf{trace}(YX) = \mathsf{trace}(XY) = \mathsf{trace}(XY) = 0$$

so $\mathfrak{sl}_n(\mathbb{F})$ is indeed a Lie algebra, though it is not an algebra.

We refer to $\mathfrak{sl}_n(\mathbb{F})$ as the *special linear Lie algebra*.

(B) Suppose $n = 2m+1$ is odd. Let $(\cdot)^\top$ be the usual matrix transpose and set

$$\mathfrak{o}_n(\mathbb{F}) \overset{\text{def}}{=} \left\{ \begin{bmatrix} 0 & A & B \\ -A^\top & M & N \\ -B^\top & P & -M^\top \end{bmatrix} : A, B \in \mathbb{F}^m,\ M, N, P \in \mathfrak{gl}_m(\mathbb{F}),\ N = -N^\top,\ P = -P^\top \right\}.$$

We call $\mathfrak{o}_n(\mathbb{F})$ the *odd orthogonal Lie algebra*.

(C) Suppose $n = 2m$ is even. Let

$$\mathfrak{sp}_n(\mathbb{F}) \overset{\text{def}}{=} \left\{ \begin{bmatrix} M & N \\ P & -M^\top \end{bmatrix} : M, N, P \in \mathfrak{gl}_m(\mathbb{F}),\ N = N^\top,\ P = P^\top \right\}.$$

We call $\mathfrak{sp}_n(\mathbb{F})$ the *symplectic Lie algebra*.

(D) Suppose $n = 2m$ is even. Let

$$\mathfrak{o}_n(\mathbb{F}) \overset{\text{def}}{=} \left\{ \begin{bmatrix} M & N \\ P & -M^\top \end{bmatrix} : M, N, P \in \mathfrak{gl}_m(\mathbb{F}),\ N = -N^\top,\ P = -P^\top \right\}.$$

We call $\mathfrak{sp}_n(\mathbb{F})$ the *even orthogonal Lie algebra*.

These examples make up the *classical Lie algebras* of types A, B, C, and D.

It is a nontrivial exercise to check that there subspaces are in fact Lie algebras. At the end of this lecture, we will see that each example is a special case of a general construction that takes care of this for us.

5 Properties of the Lie bracket

Let $L \subseteq \mathfrak{gl}(V)$ be a Lie algebra.

This is a subspace with $[X,Y] = XY - YX \in L$ for all $X, Y \in L$.

The operation $[\cdot,\cdot]$ is called the *Lie bracket*.

Here are its key properties. Let $a, b \in \mathbb{F}$ and $X, Y, Z \in L$.

(1) The Lie bracket is *bilinear* in the sense that

$$[aX + bY, Z] = a[X,Z] + b[Y,Z] \quad \text{and} \quad [X, aY + bZ] = a[X,Y] = b[X,Z].$$

(2) The Lie bracket is *alternating* in the sense that

$$[X, X] = 0.$$

(3) The previous two properties imply that the Lie bracket is *skew-symmetric* in the sense

$$[X, Y] = -[Y, X].$$

This follows since

$$0 = [X+Y, X+Y] = [X, X+Y] + [Y, X+Y] = [X, X] + [X, Y] + [Y, X] + [Y, Y] = [X, Y] + [Y, X].$$

(4) Let $\mathsf{ad}_X = [X, \cdot]$ be the linear map $L \to L$ defined by $\mathsf{ad}_X(Y) = [X, Y]$. Then the *Jacobi identity*

$$\mathsf{ad}_{[X,Y]} = [\mathsf{ad}_X, \mathsf{ad}_Y]$$

holds when we set $[\mathsf{ad}_X, \mathsf{ad}_Y] = \mathsf{ad}_X \mathsf{ad}_Y - \mathsf{ad}_Y \mathsf{ad}_X$.

More generally, whenever f and g are objects that can be multiplied or composed in either order, and $[f, g]$ has not been given some other meaning, we define

$$[f, g] = fg - gf.$$

Proof of Jacobi identity. Both $\mathsf{ad}_{[X,Y]}$ and $[\mathsf{ad}_X, \mathsf{ad}_Y]$ are maps $L \to L$.

To check that they are equal, we check that they have the same value at each $Z \in L$.

Specifically, we have

$$\mathsf{ad}_{[X,Y]}(Z) = [[X, Y], Z] = [XY - YX, Z] = XYZ - YXZ - ZXY + ZYX.$$

On the other hand,

$$\begin{aligned}
[\mathsf{ad}_X, \mathsf{ad}_Y](Z) &= \mathsf{ad}_X \mathsf{ad}_Y(Z) - \mathsf{ad}_Y \mathsf{ad}_X(Z) \\
&= \mathsf{ad}_X(YZ - ZY) - \mathsf{ad}_Y(XZ - ZX) \\
&= XYZ - XZY - YZX + ZYX - YXZ + YZX + XZY - ZXY. \\
&= XYZ + ZYX - YXZ - ZXY.
\end{aligned}$$

As desired, these two expressions are equal. □

Since $L \subseteq \mathfrak{gl}(V)$ is a vector space, we can form another Lie algebra $\mathfrak{gl}(L)$.

The Jacobi identity says that the operation $\mathsf{ad} : L \to \mathfrak{gl}(L)$ is a *Lie algebra homomorphism*.

However, for this claim to be meaningful we still need to define what a homomorphism is.

6 Abstract definition of a Lie algebra

Suppose L is any $\mathbb{F}$-vector space with a map $[\cdot, \cdot] : L \times L \to L$, to be called the *Lie bracket*.

For each $X \in L$ define $\mathsf{ad}_X \in \mathfrak{gl}(L)$ by $\mathsf{ad}_X(Y) = [X, Y]$.

For $f, g \in \mathfrak{gl}(L)$ let $[f, g] = fg - gf = f \circ g - g \circ f$.

Remark. Notice that we are reusing the symbol $[\cdot, \cdot]$ to mean two different maps

$$L \times L \to L \quad \text{and} \quad \mathfrak{gl}(L) \times \mathfrak{gl}(L) \to \mathfrak{gl}(L).$$

This might be slightly confusing, but we can always use context to determine which definition applies.

Definition. The vector space L is a *Lie algebra* with respect to $[\cdot,\cdot]$ if the following conditions hold:

(L1) the Lie bracket is bilinear,

(L2) the Lie bracket is alternating (meaning $[X,X]=0$), and

(L3) the Lie bracket satisfies the Jacobi identity $\mathsf{ad}_{[X,Y]} = [\mathsf{ad}_X, \mathsf{ad}_Y]$.

Remark. The space L may be infinite-dimensional.

However, we will rarely consider this case, since the theory is much more involved.

Unless stated otherwise, all Lie algebras L in this course are assumed to have $\dim L < \infty$.

As in the previous section, axioms (L1) and (L2) imply that

$$[X,Y] = -[Y,X]$$

for all $X,Y \in L$. This skew-symmetric property is equivalent to (L2) if and only if $\mathrm{char}(\mathbb{F}) \neq 2$.

The Jacbi identity (L3) can be restated in many ways. For example, it is equivalent to requiring

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \quad \text{for all } X,Y,Z \in L.$$

Usually a given vector space has only one natural Lie algebra structure and so we reuse the symbol $[\cdot,\cdot]$ to denote the Lie bracket for any Lie algebra. This is similar to how in group theory we write all group products as concatenation rather than introducing different symbols for various binary operations

Definition. A *Lie subalgebra* of L is a subspace $K \subseteq L$ with $[X,Y] \in K$ for all $X,Y \in K$.

Definition. A linear map $\phi : L \to K$ between Lie algebras (defined over the same field $\mathbb{F}$) is a *Lie algebra morphism* or *Lie algebra homomorphism* if

$$\phi([X,Y]) = [\phi(X), \phi(Y)] \quad \text{for all } X,Y \in L.$$

A Lie algebra morphism is an *isomorphism* if it is a bijection.

7 Abstract examples

Let V be a finite-dimensional $\mathbb{F}$-vector space.

Then the trace map $\mathsf{trace} : \mathfrak{gl}(V) \to \mathbb{F}$ is well-defined and linear.

Given $X \in \mathfrak{gl}(V)$, the trace can be computed by choosing any basis $v_1, v_2, \ldots, v_n$ for V.

If we define $X_{ij} \in \mathbb{F}$ such that $X(v_j) = \sum_{i=1}^n X_{ij} v_i$ then $\mathsf{trace}(X) = X_{11} + X_{22} + \cdots + X_{nn}$.

This definition does not depend on the choice of basis for V.

Definition. The *special linear Lie algebra* of V is

$$\mathfrak{sl}(V) = \{X \in \mathfrak{gl}(V) : \mathsf{trace}(X) = 0\}.$$

This subspace is a Lie subalgebra of $\mathfrak{gl}(V)$ by the same argument as in the $\mathfrak{gl}_n(\mathbb{F})$ case earlier.

Now suppose $B : V \times V \to V$ is a bilinear form.

If $V = \mathbb{F}^n$ then every such B has the formula $B(x,y) = x^\top M y$ for some $n \times n$ matrix M.

Proposition. The subspace

$$L_B \overset{\text{def}}{=} \{X \in \mathfrak{gl}(V) : B(Xu, v) = -B(u, Xv) \text{ for all } u, v \in V\}$$

is a Lie subalgebra of $\mathfrak{gl}(V)$.

Proof. If $X, Y \in L_B$ then $[X, Y] \in L_B$ since

$$\begin{aligned} B([X,Y]u, v) &= B(XYu, v) - B(YXuv,) \\ &= -B(Yu, Xv) + B(Xu, Yv) \\ &= B(u, YXv) - B(u, XYv) = -B(u, [X,Y]v). \end{aligned}$$

□

Assume $V = \mathbb{F}^n$. Let I_n be the $n \times n$ identity matrix. Recall that we have a canonical way to identify $\mathfrak{gl}(\mathbb{F}^n) = \mathfrak{gl}_n(\mathbb{F})$, namely, by representing each linear map $\mathbb{F}^n \to \mathbb{F}^n$ by its standard matrix.

Example. One can check that $\mathfrak{o}_n(\mathbb{F}) = L_B$ if B is the bilinear form corresponding to the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_m \\ 0 & I_m & 0 \end{bmatrix} \text{ when } n = 2m+1 \text{ is odd} \quad \text{or} \quad M = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} \text{ when } n = 2m \text{ is even.}$$

Example. When $n = 2m$ is even, we have $\mathfrak{sp}_n(\mathbb{F}) = L_B$ if B is the bilinear form corresponding to

$$M = \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} \text{ when } n = 2m \text{ is even.}$$