1 Lie algebras

Continuing from last time: suppose $\mathbb{F}$ is a field and L is an $\mathbb{F}$-vector space with a map $[\cdot,\cdot] : L \times L \to L$.

For $X \in L$ let $\mathsf{ad}_X = [X,\cdot]$ be the map $L \to L$ defined by $\mathsf{ad}_X(Y) = [X,Y]$.

If f and g are objects we can multiply and subtract, and $[f,g]$ is unspecified, then we define $[f,g] = fg - gf$.

Definition. The vector space L is a *Lie algebra* with respect to $[\cdot,\cdot]$ if the following conditions hold:

(L1) the Lie bracket is **bilinear**,

(L2) the Lie bracket is **alternating** (meaning $[X,X] = 0$), and

(L3) the Lie bracket satisfies the **Jacobi identity** $\mathsf{ad}_{[X,Y]} = \mathsf{ad}_X\mathsf{ad}_Y - \mathsf{ad}_Y\mathsf{ad}_X = [\mathsf{ad}_X, \mathsf{ad}_Y]$.

2 Algebras and derivations

Here are some more examples of Lie algebras.

Suppose A is an $\mathbb{F}$-algebra, meaning an $\mathbb{F}$-vector space with a bilinear multiplication map.

When the multiplication for A is *associative*, meaning that

$$a(bc) = (ab)c \quad \text{for all } a, b, c \in A,$$

then the vector space $\mathfrak{gl}(A)$ of all linear maps $A \to A$ is a Lie algebra for the Lie bracket

$$[X,Y] = XY - YX.$$

Checking the Jacobi identity requires associativity.

Let $\mathsf{Der}(A) \subseteq \mathfrak{gl}(A)$ be the subspace of linear maps $\delta : A \to A$ such that

$$\delta(ab) = a\delta(b) + \delta(a)b$$

for all $a, b \in A$. The elements of $\mathsf{Der}(A)$ are called *derivations*.

Example. Suppose $\mathbb{F} = \mathbb{R}$ and $A = C^\infty(\mathbb{R})$ is the commutative algebra of infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$. The product rule from calculus tells us that the usual derivative operator $\frac{d}{dx} \in \mathsf{Der}(A)$.

Example. The Lie bracket makes any Lie algebra L into a non-associative algebra.

The Jacobi identity is a substitute for associativity. It implies that $\mathsf{ad}_X \in \mathsf{Der}(L)$ for all $X \in L$ since

$$\mathsf{ad}_X([Y,Z]) = \mathsf{ad}_X\mathsf{ad}_Y(Z) = \mathsf{ad}_{[X,Y]}(Z) + \mathsf{ad}_Y\mathsf{ad}_X(Z) = [\mathsf{ad}_X(Y), Z] + [Y, \mathsf{ad}_X(Z)].$$

Proposition. The vector space $\mathsf{Der}(A)$ is a Lie subalgebra of $\mathfrak{gl}(A)$.

The proof is left as an exercise.

3 Group theory and Lie theory

Many terms in group theory reappear in the study of Lie algebras with different (but parallel) meanings.

Suppose L is a Lie algebra and G is a group. Given $X \in L$, and $g \in G$, let

$$\mathsf{ad}_X : Y \mapsto [X,Y] \quad \text{and} \quad \mathsf{Ad}_g : h \mapsto ghg^{-1}.$$

Then $\mathsf{ad}_X \in \mathfrak{gl}(L)$ and $\mathsf{Ad}_g \in \mathrm{Aut}(G)$ is a group automorphism. More strongly,

$$\mathsf{ad}_X \in \mathsf{Der}(L) \subseteq \mathfrak{gl}(L) \quad \text{and} \quad \mathsf{Ad}_g \in \mathrm{Inn}(G) \subseteq \mathrm{Aut}(G).$$

Normal subgroups versus ideals:

- A *normal subgroup* of G is a subgroup $N \subseteq G$ with $\mathsf{Ad}_g(N) \subseteq N$ for all $g \in G$.
- An *ideal* of L is a subspace $I \subseteq L$ with $\mathsf{ad}_X(I) \subseteq I$ for all $X \in L$.

  Every ideal is automatically a Lie subalgebra.

  Examples: 0 and L are always ideals of L. The Lie algebra of strictly upper-triangular matrices $\mathfrak{n}_n(\mathbb{F})$ is an ideal in the Lie algebra of all upper-triangular matrices $\mathfrak{t}_n(\mathbb{F})$.

Centers:

- The *center* of a group G is the normal subgroup $Z(G) = \{h \in G : \mathsf{Ad}_g(h) = h \text{ for all } g \in G\}$.
- The *center* of a Lie algebra L is the ideal $Z(L) = \{A \in L : \mathsf{ad}_X(A) = 0 \text{ for all } X \in L\}$.

  Notice that $Z(\mathfrak{gl}_n(\mathbb{F}))$ is **not** the Lie algebra of diagonal matrices $\mathfrak{d}_n(\mathbb{F})$ but is instead the 1-dimensional Lie algebra of *scalar matrices* spanned by the identity matrix.

Quotients:

- The power set of a group G is a monoid with unit $\{1\}$ and product $S \cdot T \overset{\text{def}}{=} \{gh : g \in S \text{ and } h \in T\}$.

  This product is associative, but not every subset is invertible.

  Every subgroup $H \subseteq G$ is *idempotent* in this monoid in the sense that $H \cdot H = H$.

  For a normal subgroup $N \subseteq G$ the set of left cosets $G/N = \{gN : g \in G\}$ is a submonoid of the power set of G that actually forms a group, since for any $g, h \in G$ we have the product formula

$$(gN)(hN) = gh \cdot \underbrace{h^{-1}Nh}_{=N} \cdot N = gh \cdot N \cdot N = (gh)N.$$

  The unit of the *quotient group* G/N is the subgroup N itself.

- Given an ideal $I \subseteq L$ the quotient space $L/I = \{X + I : X \in L\}$ is the set of cosets of I in L.

  This is a Lie algebra for the modified Lie bracket

$$[X + I, Y + I] \overset{\text{def}}{=} [X, Y] + I \quad \text{for } X, Y \in L.$$

  This Lie bracket is well-defined (meaning its value depends only on the cosets $X + I$ and $Y + I$ rather than the representatives X and Y) because I is an ideal.

Remark. Suppose $I, J \subseteq L$ are both ideals of a Lie algebra.

(a) Then the vector space sum $I + J$ is a Lie subalgebra, in which J is an ideal.

  The quotients $(I + J)/J \cong I/(I \cap J)$ are isomorphic as Lie algebras.

(b) If $I \subseteq J$ then J/I is an ideal of L/I and there is an isomorphism $(L/I)/(J/I) \cong L/J$.

Kernels:

- The *kernel* of a group homomorphism $\phi : G \to H$ is the subset $\ker(\phi) = \{g \in G : \phi(g) = 1\}$.

  This subset is a normal subgroup, and if ϕ is surjective then $G/\ker(\phi) \cong H$.

- The *kernel* of a Lie algebra homomorphism $\phi : L \to K$ is the subspace $\ker(\phi) = \{X \in L : \phi(X) = 0\}$.
  This subspace is an ideal, and if ϕ is surjective then $L/\ker(\phi) \cong K$.

Derived subalgebras and subgroups:

- The *derived subgroup* of $[G, G]$ is the subgroup generated by all commutators $ghg^{-1}h^{-1}$ for $g, h \in G$.
  This means $[G, G]$ is the intersection of all subgroups $H \subseteq G$ containing all commutators $ghg^{-1}h^{-1}$.
  Instructive exercise: check that $[G, G]$ is a normal subgroup.
- The *derived Lie subalgebra* $[L, L]$ is the subspace of L spanned by all brackets $[X, Y]$ for $X, Y \in L$.
  This subspace is an ideal as $[X$, a span of brackets$]$ = another span of brackets.
  Instructive exercise: check that $[\mathfrak{gl}_n(\mathbb{F}), \mathfrak{gl}_n(\mathbb{F})] = \mathfrak{sl}_n(\mathbb{F})$.

Abelian Lie algebras and groups:

- A group G is *abelian* if $gh = hg$ for all $g, h \in G \quad \Leftrightarrow \quad G = Z(G) \quad \Leftrightarrow \quad [G, G] = 1$.
- A Lie algebra L is *abelian* if $[X, Y] = 0$ for all $X, Y \in L \quad \Leftrightarrow \quad L = Z(L) \quad \Leftrightarrow \quad [L, L] = 0$.

Simple Lie algebras and groups:

- A group is *simple* if it is nontrivial with no nontrivial, proper normal subgroups.
  All groups of prime order are simple, and these are all abelian.
- A Lie algebra is *simple* if it is non-abelian with no nonzero, proper ideals.
  The analogy here is that abelian Lie algebras are like trivial groups (rather than abelian groups).
  Notice that in an abelian Lie algebra every subspace is an ideal.
  So the only abelian Lie algebras with no nonzero, proper ideals are 1-dimensional.
  These are excluded from being simple to make some classification theorems easier to state.

4 Example: a simple Lie algebra

Recall that $\mathfrak{sl}_2(\mathbb{F}) = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} : a, b, c \in \mathbb{F} \right\}$ is a Lie algebra with Lie bracket $[X, Y] = XY - YX$.

One basis for this Lie algebra consists of the elements

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The Lie brackets in this basis are $[E, E] = [F, F] = [H, H] = 0$ along with

$$[E, F] = -[F, E] = H, \qquad [H, E] = -[E, H] = 2E, \qquad \text{and} \qquad [H, F] = -[F, H] = -2F.$$

Notice that E, F, and H are eigenvectors for ad_H with respective eigenvalues 2, -2, and 0.

Thus ad_H is a diagonalizable linear map $\mathfrak{sl}_2(\mathbb{F}) \to \mathfrak{sl}_2(\mathbb{F})$.

Lemma. Assume $\mathrm{char}(\mathbb{F}) \neq 2$ and $I \subseteq \mathfrak{sl}_2(\mathbb{F})$ is an ideal. If $E \in I$ or $F \in I$ or $H \in I$ then $I = \mathfrak{sl}_2(\mathbb{F})$.

Proof. Suppose $E \in I$. Then $H = [-F, E] \in I$ so also $F = [-\frac{1}{2}F, H] \in I$ since $\mathrm{char}(\mathbb{F}) \neq 2$.

It follow similarly that if $F \in I$ then also $H, E \in I$ and if $H \in I$ then also $E, F \in I$.

Hence if $E \in I$ or $F \in I$ or $H \in I$ then $I = \mathfrak{sl}_2(\mathbb{F})$ since ideals are subspaces. □

Proposition. Assume $\mathrm{char}(\mathbb{F}) \neq 2$. Then $\mathfrak{sl}_2(\mathbb{F})$ is simple.

Proof. Suppose $I \subseteq \mathfrak{sl}_2(\mathbb{F})$ is a nonzero ideal.

Then there is an element $0 \neq g = aE + bF + cH \in I$ for some coefficients $a, b, c \in \mathbb{F}$. Observe that

$$[E, [E, g]] = [E, [bH - 2cE]] = -2bE \in I \quad \text{and} \quad [F, [F, g]] = [F, aH - 2cF] = -2aF \in I.$$

If $b \neq 0$ then $E \in I$ as $\mathrm{char}(\mathbb{F}) \neq 2$ so $I = \mathfrak{sl}_2(\mathbb{F})$ by the lemma.

If $a \neq 0$ then $F \in I$ as $\mathrm{char}(\mathbb{F}) \neq 2$, so $I = \mathfrak{sl}_2(\mathbb{F})$ by the lemma.

If $a = b = 0$ then we must have $c \neq 0$ but then $H \in I$ so again $I = \mathfrak{sl}_2(\mathbb{F})$ by the lemma.

Thus any nonzero ideal I is equal to $\mathfrak{sl}_2(\mathbb{F})$, so the non-abelian Lie algebra $\mathfrak{sl}_2(\mathbb{F})$ is simple. □

5 Normalizers and centralizers

Suppose L is a Lie algebra with a Lie subalgebra K.

Notice that if $X, Y, Z \in L$ then the Jacobi identity tells us that

$$\mathsf{ad}_{[X,Y]}(Z) = \mathsf{ad}_X \mathsf{ad}_Y(Z) - \mathsf{ad}_Y \mathsf{ad}_X(Z). \qquad (*)$$

The *normalizer* of K is the subspace $N_L(K) = \{X \in L : \mathsf{ad}_X(K) \subseteq K\}$.

The *centralizer* of K is the subspace $C_L(K) = \{X \in L : \mathsf{ad}_X(K) = 0\}$.

Proposition. Both $C_L(K) \subseteq N_L(K)$ are Lie subalgebras of L.

Proposition. The Lie subalgebras K and $C_L(K)$ are both ideals of $N_L(K)$.

Both propositions are consequences of (*). We leave checking the details as an exercise.

By definition, any subalgebra of L that contains K as an ideal is a subset of $N_L(K)$.

Thus $N_L(K)$ is the largest Lie subalgebra of L that contains K as an ideal.

6 Representations of Lie algebras

Let L be a Lie algebra. A *representation* of L is a Lie algebra morphism

$$\phi : L \to \mathfrak{gl}(V)$$

for some vector space V (defined over the same field as L). The vector space V can be infinite-dimensional.

The only abstract algebra we can do efficiently is matrix algebra, and knowing a representation of a Lie algebra lets us convert calculations in the Lie algebra to matrix algebra.

This is one reason why studying representations of Lie algebras is fundamental.

The existence of at least one representation of L is implicit in the definition of a Lie algebra:

Definition. The operation $X \mapsto \mathsf{ad}_X$ is always a morphism $L \to \mathfrak{gl}(L)$ by the Jacobi identity.

This is called the *adjoint representation* of L.

We denote the adjoint representation by ad. This gives three ways of writing the same thing:

$$\mathsf{ad}(X) = \mathsf{ad}_X = [X, \cdot] \in \mathfrak{gl}(L).$$

The image $\mathsf{ad}(L)$ of the adjoint representation is usually a proper Lie subalgebra of $\mathfrak{gl}(L)$.

In fact, the Jacobi identity implies that $\mathsf{ad}(L) \subseteq \mathsf{Der}(L)$.

The most useful representations $\phi : L \to \mathfrak{gl}(V)$ are ones that are injective.

Such representations are sometimes called *faithful*. A faithful representation determines an isomorphism from a Lie algebra to a subalgebra of a general linear Lie algebra.

Proposition. If $Z(L) = 0$ then ad is faithful so L is isomorphic to a Lie subalgebra of $\mathfrak{gl}(L)$.

Proof. We have $Z(L) = \ker(\mathsf{ad})$ so $L/Z(L) \cong \mathsf{ad}(L)$. If $Z(L) = 0$ then $L \cong L/Z(L) \cong \mathsf{ad}(L) \subseteq \mathfrak{gl}(L)$. $\square$

We always have $Z(L) = 0$ when L is simple, as $Z(L)$ is an ideal and simple Lie algebras are not abelian.

Thus every finite-dimensional simple Lie algebra can be embedded in $\mathfrak{gl}_n(\mathbb{F})$ for some n.

This previous result extends all finite-dimensional Lie algebras, but the proof in much more involved.

Theorem (Ado, 1935, for fields of characteristic zero; Iwasawa, 1948, for all fields)**.** Every Lie algebra L with $\dim(L) < \infty$ is isomorphic to a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{F})$ for some positive integer n.