1 Solvable Lie algebras

Continue our setup from last time.

Let $\mathbb{F}$ be a field. When V is an $\mathbb{F}$-vector space let $\mathfrak{gl}(V)$ be the vector space of linear maps $V \to V$.

For any maps $f, g \in \mathfrak{gl}(V)$ let $[f, g] = fg - gf$.

Let L be a Lie algebra over $\mathbb{F}$, meaning an $\mathbb{F}$-vector space with an alternating, bilinear map

$$[\cdot,\cdot] : L \times L \to L$$

satisfying the Jacobi identity $[[X, Y], Z] = [X, [Y, Z]] - [Y, [X, Z]]$ for all $X, Y, Z \in L$.

Equivalently, this means that $\mathsf{ad}_{[X,Y]} = [\mathsf{ad}_X, \mathsf{ad}_Y]$ where we write $\mathsf{ad}_X = [X, \cdot] \in \mathfrak{gl}(L)$ for $X \in L$.

For any subsets $I, J \subseteq L$ let $[I, J] = \mathbb{F}\text{-span}\{[X, Y] : X \in I \text{ and } Y \in J\}$.

The *derived subalgebra* of L is $[L, L]$. This is a Lie subalgebra.

Some new notation: inductively define

$$L^{(0)} = L, \qquad L^{(1)} = [L, L], \qquad \text{and} \qquad L^{(n+1)} = [L^{(n)}, L^{(n)}] \text{ for } n \geq 1.$$

Definition. A Lie algebra L is *solvable* if $L^{(n)} = 0$ for some $n \geq 0$.

If L is solvable then so is every subalgebra $L^{(n)}$.

Conversely, if any $L^{(n)}$ is solvable then L is solvable.

Example. Recall that $\mathfrak{gl}_n(\mathbb{F})$ is the Lie algebra $n \times n$ matrices over $\mathbb{F}$, with Lie bracket $[X, Y] = XY - YX$.

Write $\mathfrak{t}_n(\mathbb{F}) \subseteq \mathfrak{gl}_n(\mathbb{F})$ for the Lie subalgebra of upper-triangular matrices. (Equality holds if $n = 1$.)

Also let $\mathfrak{n}_n(\mathbb{F}) \subsetneq \mathfrak{t}_n(\mathbb{F})$ be the Lie subalgebra of strictly upper-triangular matrices. Then

$$[\mathfrak{t}_n(\mathbb{F}), \mathfrak{t}_n(\mathbb{F})] = \mathfrak{n}_n(\mathbb{F}).$$

Any product of at least n elements of $\mathfrak{n}_n(\mathbb{F})$ is zero: strictly upper-triangular matrices are *nilpotent*.

Therefore $\mathfrak{n}_n(\mathbb{F})$ is solvable, since if $L = \mathfrak{n}_n(\mathbb{F})$ and $m \geq 1$ then the brackets spanning $L^{(m)}$ can be expanded into linear combinations of terms which are each products of 2^m elements of $\mathfrak{n}_n(\mathbb{F})$, which are all zero is m is large enough. For example,

$$\begin{aligned}[][[A, B], [C, D]] &= [AB - BA, CD - DC] \\ &= ABCD - ABDC - DACD + BADC - CDAB + CDBA + DCAB - DCBA \in L^{(2)}.\end{aligned}$$

This implies that $\mathfrak{t}_n(\mathbb{F})$ is also solvable.

Recall that a Lie algebra morphism $\phi : L \to K$ is a linear map with $\phi([X, Y]) = [\phi(X), \phi(Y)]$.

Proposition. If L is solvable, then all Lie subalgebras and all homomorphic images of L are solvable.

Proof. If $K \subseteq L$ is a Lie subalgebra then $K^{(n)} \subseteq L^{(n)}$, so if $L^{(n)} = 0$ then $K^{(n)} = 0$.

If $\phi : L \to K$ is a Lie algebra morphism then $\phi(L)^{(n)} = \phi(L^{(n)})$, so if $L^{(n)} = 0$ then $\phi(L)^{(n)} = 0$. □

Proposition. Suppose $I \subseteq L$ is an ideal. Then L is solvable if and only if I and L/I are both solvable.

Proof. Assume L is solvable.

Then the Lie subalgebra I is solvable and L/I is a solvable as it is the image of the morphism $L \to L/I$.

Assume instead that I and L/I are both solvable.

In this case $L^{(n)} \subseteq I$ for the value of $n \gg 0$ with $(L/I)^{(n)} = 0$, and $I^{(m)} = 0$ for some $m \gg 0$.

Then we have $L^{(m+n)} \subseteq I^{(m)} = 0$ so L is solvable. □

Proposition. If $I, J \subseteq L$ are both solvable ideals then the Lie subalgebra $I + J$ is solvable.

Proof. In this case $(I+J)/I \cong J/(I \cap J)$ is solvable as the second quotient is a homomorphic image of J.

The result thus follows from the previous proposition, replacing L by $I + J$. □

Corollary. If $\dim(L) < \infty$ then L has a unique maximal solvable ideal.

Proof. The zero ideal is solvable, so the set of solvable ideals in L is nonempty.

Since $\dim(L) < \infty$ there must be at least one solvable ideal of L that is maximal under inclusion.

Suppose I and J are two such maximal solvable ideals.

Then $I+J$ is solvable and contains both I and J, so the maximality hypothesis implies $I = I+J = J$. □

For the rest of this section assume $\dim(L) < \infty$.

Let $\mathsf{Rad}(L)$ denote the unique maximal solvable ideal of L, to be called the *radical* of L.

Notice that L is solvable if and only if $\mathsf{Rad}(L) = L$.

Definition. A finite-dimensional Lie algebra L is *semisimple* if $\mathsf{Rad}(L) = 0$.

We have $\mathsf{Rad}(L) = 0$ if and only if L has no nonzero solvable ideals.

A semisimple Lie algebra is in this sense as far away from solvable as possible.

The word "semisimple" usually is a synonym for "direct sum of simple objects."

This will be the case for Lie algebras too.

We will prove later that L is semsimple if and only if L is a direct sum of simple Lie algebras.

Proposition. The quotient Lie algebra $L/\mathsf{Rad}(L)$ is always semisimple.

Proof. Each nonzero ideal in $L/\mathsf{Rad}(L)$ has the form $I/\mathsf{Rad}(L)$ for an ideal $I \subseteq L$ which properly contains $\mathsf{Rad}(L)$. Such an ideal is not solvable, so $I/\mathsf{Rad}(L)$ must not be solvable either (by earlier propsition). □

Proposition. If L is simple then L is semsimple and not solvable.

Proof. Assume L is simple. Then L is not abelian so $[L, L] \neq 0$.

But $[L, L]$ is an ideal of L, so we must have $[L, L] = L$.

Hence $L^{(n)} = L$ for all n so L is not solvable.

But this means that $\mathsf{Rad}(L)$ is a proper ideal, so we must have $\mathsf{Rad}(L) = 0$. □

2 Nilpotent Lie algebras

Continue to let L be a Lie algebra. As a variant of our earlier notation, inductively define

$$L^0 = L^{(0)} = L, \quad L^1 = L^{(1)} = [L, L], \quad \text{and} \quad L^{n+1} = [L, L^n] = [L^n, L] \quad \text{for integers } n \geq 1.$$

Definition. A Lie algebra L is *nilpotent* if $L^n = 0$ for some $n \geq 0$.

Notice that $L^{(n)} \subseteq L^n$ for all n.

Therefore if $L^n = 0$ then $L^{(n)} = 0$, so any nilpotent Lie algebra is solvable.

But not every solvable Lie algebra is nilpotent.

Example. If $L = \mathfrak{t}_n(\mathbb{F})$ is the Lie algebra of upper-triangular matrices then $L^n = \mathfrak{n}_n(\mathbb{F})$ for all $n \geq 1$. Thus $\mathfrak{t}_n(\mathbb{F})$ is solvable but not nilpotent.

Proposition. If L is nilpotent, then all Lie subalgebras and homomorphic images of L are nilpotent.

Proof. If $K \subseteq L$ is a Lie subalgebra then $K^n \subseteq L^n$, so if $L^n = 0$ then $K^n = 0$.

If $\phi : L \to K$ is a Lie algebra morphism then $\phi(L)^n = \phi(L^n$, so if $L^n = 0$ then $\phi(L)^n = 0$. □

Recall that the center of L is the ideal $Z(L) = \{X \in L : \mathsf{ad}_X = 0\}$.

Proposition. If $L/Z(L)$ is nilpotent then L is nilpotent.

Proof. In this case we must have $L^n \subseteq Z(L)$ for some $n \gg 0$ and then $L^{n+1} \subseteq [L, Z(L)] = 0$. □

Proposition. If L is nilpotent and $L \neq 0$ then $Z(L) \neq 0$.

Proof. In this case if $L^n \neq 0$ but $L^{n+1} = 0$ then $0 \neq L^n \subseteq Z(L)$. □

Proposition. A Lie algebra L is nilpotent if and only if for some sufficiently large $n \gg 0$ we have

$$\mathsf{ad}_{X_1} \mathsf{ad}_{X_2} \cdots \mathsf{ad}_{X_n} = 0 \in \mathfrak{gl}(L) \quad \text{for all } X_1, X_2, \ldots, X_n \in L. \tag{*}$$

Proof. Observe that L^n is spanned by elements of the form $\mathsf{ad}_{X_1} \mathsf{ad}_{X_2} \cdots \mathsf{ad}_{X_n}(Y)$. □

An element $X \in L$ is ad-*nilpotent* if ad_X is a nilpotent linear map, meaning $(\mathsf{ad}_X)^n = 0$ for some $n \geq 0$. Taking $X_1 = X_2 = \cdots = X_n$ in (*) implies that:

Corollary. If L is nilpotent then all of its elements are ad-nilpotent.

It is not obvious that if all elements of L are ad-nilpotent then the stronger condition (*) holds, which would imply that L is nilpotent. In the next section we will see that this does hold if $\dim(L) < \infty$.

3 Engel's theorem

Let V be any $\mathbb{F}$-vector space.

Lemma. If $X \in \mathfrak{gl}(V)$ is nilpotent (meaning $X^n = 0$ for some $n \gg 0$) then $\mathsf{ad}_X \in \mathfrak{gl}(\mathfrak{gl}(V))$ is nilpotent.

Proof. This lemma holds even if V is infinite-dimensional.

Given $X \in \mathfrak{gl}(V)$, define λ_X and ρ_X to be the linear maps $\mathfrak{gl}(V) \to \mathfrak{gl}(V)$ with the formulas

$$\lambda_X(Y) = XY \quad \text{and} \quad \rho_X(Y) = YX.$$

Because $\mathfrak{gl}(V)$ is an associative algebra, λ_X and ρ_X commute with each other.

When X is nilpotent, the maps λ_X and ρ_X are both nilpotent.

But $\mathsf{ad}_X = \lambda_x - \rho_X$ and any finite linear combination of nilpotent linear maps is nilpotent.

In particular, if $n \gg 0$ is large enough that $(\lambda_X)^n = (\rho_X)^n = 0$ then

$$(\mathsf{ad}_X)^{2n} = (\lambda_X - \rho_X)^{2n} = \sum_{k=0}^{2n} \binom{2n}{n} (-1)^{2n-k} (\lambda_X)^k (\rho_X)^{2n-k} = 0.$$

since either $k \geq n$ or $2n - k \geq n$. □

Now assume $\dim(V) < \infty$ and $L \subseteq \mathfrak{gl}(V)$ is a Lie subalgebra.

By Ado's theorem, every finite-dimensional Lie algebra arises in this way.

Because we are explicitly embedding L inside $\mathfrak{gl}(V)$, we can multiply two elements of L, although the product XY for $X, Y \in L$ is not guaranteed to belong to L (we only know that $XY - YX \in L$).

This means we can form $X^n \in \mathfrak{gl}(V)$ for any $X \in L$ and $n \geq 0$.

(This would not make sense for an abstract Lie algebra that only has a Lie bracket.)

We say that $X \in L$ is *nilpotent* if $X^n = 0$ for some $n \gg 0$.

Theorem. Suppose $\dim(V) < \infty$ and $L \subseteq \mathfrak{gl}(V)$ is a Lie subalgebra.

If every element of L is nilpotent, then there exists a vector $0 \neq v \in V$ with $Xv = 0$ for all $X \in L$.

Proof. Any nilpotent linear map $X \in \mathfrak{gl}(V)$ has an eigenvector with eigenvalue zero.

If $\dim(L) \leq 1$ then we can just take $v \in V$ to be any 0-eigenvector of any $0 \neq X \in L$.

Assume $\dim(L) > 1$ and let $K \subsetneq L$ be a maximal proper subalgebra. By induction on dimension (replacing L and V by $\mathsf{ad}(K)$ and L/K) some $X \in L$ has $X \notin K$ and $[Y, X] \in K$ for all $Y \in K$.

This means that $K \subsetneq N_L(K)$ because $X \in N_L(K)$ but $X \notin K$.

Then the maximality of K implies that $L = N_L(K)$ so K is actually an ideal of L.

Choose $Z \in L$ with $Z \notin K$. Since K is an ideal, the direct sum $K \oplus \mathbb{F}Z$ is a Lie subalgebra of L properly containing K, so by maximality we must have $L = K \oplus \mathbb{F}Z$ and $\dim(L) = \dim(K) + 1$.

By induction on $\dim(L)$, the subspace $W = \{v \in V : Yv = 0 \text{ for all } Y \in K\}$ is nonzero.

Every $X \in L$ restricts to a nilpotent linear map $W \to W$ since if $w \in W$ and $Y \in K$ then

$$YXw = X \underbrace{Yw}_{=0} - \underbrace{[X,Y]}_{\in K} w = 0.$$

In particular, Z restricts to a nilpotent linear map $W \to W$, so it has a 0-eigenvector $0 \neq v \in W$.

This vector has $Yv = 0$ for all $Y \in K$, so it also satisfies $Xv = 0$ for all $X \in L = K \oplus \mathbb{F}Z$. □

Corollary. If L is nilpotent with $\dim(L) < \infty$ and $K \subseteq L$ is an ideal then $Z(L) \cap K \neq 0$.

Proof. Apply the previous theorem with V and L replaced by K and $\mathsf{ad}(L)$ to get an element $0 \neq X \in K$ with $\mathsf{ad}_Y(X) = [Y,X] = 0$ for all $Y \in L$. This is a nonzero element of $Z(L) \cap K$. □

Theorem (Engel's theorem)**.** Suppose $0 < \dim V < \infty$ and $L \subseteq \mathfrak{gl}(V)$ is a Lie subalgebra.

Then L is nilpotent if and only if every element $X \in L$ is ad-nilpotent.

Proof. We already know that if L is nilpotent then every element $X \in L$ is ad-nilpotent

Assume conversely that every element $X \in L$ is ad-nilpotent.

Then $\mathsf{ad}(L) \subseteq \mathfrak{gl}(L)$ satisfies the conditions of the previous theorem.

Hence there exists $0 \neq X \in L$ with $\mathsf{ad}_Y(X) = [Y,X] = 0$ for all $Y \in L$.

This means that $Z(L) \neq 0$, so the quotient Lie algebra $L/Z(L)$ has smaller dimension than L.

The elements of $L/Z(L)$ are still all ad-nilpotent, so by induction $L/Z(L)$ is nilpotent.

Hence by the propositions above it follows that L is nilpotent. □

Corollary. Suppose $\dim V = n < \infty$ and $L \subseteq \mathfrak{gl}(V)$ is a Lie subalgebra whose elements are all nilpotent.

Then there is a basis of V relative to which the matrices of all elements of L are strictly upper-triangular.

Proof. It suffices to show that there is an increasing chain of vector spaces

$$0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = V$$

such that $XV_i \subseteq V_{i-1}$ for all i and $X \in L$.

Then the desired basis is obtained by choosing an element $v_i \in V_i \setminus V_{i-1}$ for each $i = 1, 2, 3, \ldots, n$.

We can construct such a chain by setting $V_1 = \mathbb{F}v$ where $0 \neq v \in V$ has $Xv = 0$ for all $X \in L$.

Then apply induction to the image of L in $\mathfrak{gl}(V/V_1)$. □

Corollary. If L is nilpotent with $\dim(L) = n < \infty$ then $\mathsf{ad}(L)$ is isomorphic to a Lie subalgebra of $\mathfrak{n}_n(\mathbb{F})$.

Proof. Apply the previous corollary with V and L replaced by L and $\mathsf{ad}(L)$. □